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A conditional limit theorem for tree-indexed random walk

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Abstract

We consider Galton–Watson trees associated with a critical offspring distribution and conditioned to have exactly n vertices. These trees are embedded in the real line by assigning spatial positions to the vertices, in such a way that the increments of the spatial positions along edges of the tree are independent variables distributed according to a symmetric probability distribution on the real line. We then condition on the event that all spatial positions are nonnegative. Under suitable assumptions on the offspring distribution and the spatial displacements, we prove that these conditioned spatial trees converge as $n \to \infty$, modulo an appropriate rescaling, towards the conditioned Brownian tree that was studied in previous work. Applications are given to asymptotics for random quadrangulations.

Keywords: Galton–Watson tree; Tree-indexed random walk; Spatial tree; Conditioned tree; Conditioned Brownian snake; Invariance principle; ISE; Well-labelled tree; Random quadrangulations

1. Introduction

The main goal of the present work is to prove an invariance principle for tree-indexed random walks on the real line which are constrained to remain on the positive side. One major motivation for this problem came from recent asymptotic results for random quadrangulations which have been established by Chassaing and Schaeffer [8].

The asymptotic behavior of Galton–Watson trees conditioned to have a large fixed progeny was investigated by Aldous [1] in connection with the so-called Continuum Random Tree (CRT).

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Precisely, under the assumption that the offspring distribution μ is critical and has finite variance $\sigma^2 > 0$, a Galton–Watson tree conditioned to have exactly n vertices, with edges rescaled by the factor $\sigma n^{-1/2}/2$, will converge in distribution, in a suitable sense, towards the CRT. A convenient way of making this convergence mathematically precise is to use the contour function of the conditioned Galton–Watson tree (cf. Fig. 1 below). Modulo a rescaling analogous to the classical Donsker theorem for random walks, this contour function converges in distribution as $n \to \infty$ towards a normalized Brownian excursion, that is a positive Brownian excursion conditioned to have duration 1 (cf. the convergence of the first components in Theorem 2.1 below). Informally we may say that the normalized Brownian excursion is the contour function of the CRT. See [14] for analogous contour descriptions of the more general Lévy trees, and [13] for a recent generalization of Aldous' theorem.

In view of various applications, and in particular in connection with the theory of superprocesses, it is interesting to combine the branching structure of the Galton–Watson tree with spatial displacements. Here we consider the simple special case where these spatial displacements are given by a one-dimensional symmetric random walk on the real line with jump distribution γ. This means that i.i.d. random variables Y_e with distribution γ are associated with the different edges of the tree, and that the spatial position U_v of a vertex v is obtained by summing the displacements Y_e corresponding to edges e that belong to the path from the root to the vertex v. The resulting object, called here a spatial tree, consists of a (random) pair $(\mathcal{T}, U)$, where $\mathcal{T}$ is a discrete (plane) tree and U is a mapping from the set of vertices of $\mathcal{T}$ into $\mathbb{R}$. In the same way as the tree $\mathcal{T}$ can be coded by its contour function, a convenient way of encoding the spatial positions is via the spatial contour function (see Section 2 for a precise definition, and Fig. 2 for an example).

From now on, we suppose that the tree $\mathcal{T}$ is a Galton–Watson tree with offspring distribution μ satisfying the above assumptions, and conditioned to have exactly n vertices, and that the spatial positions U_v are generated as explained in the preceding paragraph. We assume furthermore that μ has (small) exponential moments and that $\gamma([x, \infty)) = o(x^{-4})$ as $x \to \infty$. We denote by ρ^2 the variance of γ. Then rescaling both the edges of $\mathcal{T}$ by the factor $\sigma n^{-1/2}/2$ and the spatial displacements U_v by $\rho^{-1}(\sigma/2)^{1/2}n^{-1/4}$ will lead as $n \to \infty$ to a limiting object which is independent of μ and γ. A precise statement for this convergence is given in Theorem 2.1 below, which is taken from Janson and Marckert [19] (see [8,16,26] for similar statements under different assumptions — related results have also been obtained by Kesten [20] under other conditionings of the tree). This convergence is closely related to the approximation of superprocesses by branching particle systems: see in particular [21]. Roughly speaking, the limiting object combines the branching structure of the CRT with spatial displacements given by independent linear Brownian motions along the edges of the tree. A convenient representation of this limiting object, which is used in Theorem 2.1, is provided by the Brownian snake (see e.g. [22]). To describe this approach, let $r \in \mathbb{R}$, which will represent the initial position (the spatial position of the root) and let $\mathbf{e} = (\mathbf{e}(s), 0 \le s \le 1)$ be a normalized Brownian excursion. Let $Z^r = (Z^r(s), 0 \le s \le 1)$ be a real-valued process such that, conditionally given $\mathbf{e}$, Z^r is Gaussian with mean and covariance given by

- $E[Z^r(s)] = r$ for every $s \in [0, 1]$;
- $\mathrm{cov}(Z^r(s), Z^r(s')) = \inf_{s \le t \le s'} \mathbf{e}(t)$ for every $0 \le s \le s' \le 1$.

Informally, each time $s \in [0, 1]$ corresponds via the contour function coding to a vertex of the CRT, and $Z^r(s)$ is the spatial position of this vertex. The formula for the conditional covariance of $Z^r(s)$ and $Z^r(s')$ is then justified by the fact that $\inf_{s \le t \le s'} \mathbf{e}(t)$ is the generation of the "most recent common ancestor" to the vertices corresponding to s and s' (see the introduction to [24] for a precise version of this informal explanation). In the terminology of [22], Z^r is the terminal point process of the one-dimensional Brownian snake driven by the normalized Brownian excursion $\mathbf{e}$ and with initial point r. For simplicity, we will say here that $(\mathbf{e}, Z^r)$ is the Brownian snake with initial point r. The random measure known as one-dimensional ISE (Integrated Super-Brownian Excursion) may be defined by the formula:

$$\langle \mathcal{Z}, f\rangle = \int_0^1 \mathrm{d}s\, f(Z_s^0), \qquad f \in C_b(\mathbb{R}).$$

ISE in higher dimensions has found many applications in asymptotics for models of statistical physics: see in particular [12,17] and [18].

Our main interest in this work is to study asymptotics for the discrete spatial trees $(\mathcal{T}, U)$ conditioned on the event that spatial positions U_v remain in the positive half-line. In contrast with the situation described above, this induces an interaction between the branching structure and the spatial displacements, which makes the analysis of the model more delicate. We start from a spatial tree $(\mathcal{T}, U)$ generated as explained above. Then our main result (Theorem 2.2) states that this spatial tree conditioned on the event that $U_v \geq 0$ for every vertex v of $\mathcal{T}$ and rescaled as previously will converge in distribution as $n \to \infty$ to a (universal) limiting object. This limiting object corresponds to the conditioned Brownian snake that was studied in detail in [24]. More precisely, for every $r > 0$, let $(\overline{\mathbf{e}}^r, \overline{Z}^r)$ be distributed as the pair $(\mathbf{e}, Z^r)$ introduced above, under the conditioning that $Z^r(s) \geq 0$ for every $s \in [0, 1]$. Note that as long as $r > 0$, this conditioning involves an event of positive probability. Theorem 1.1 in [24] shows that the process $(\overline{\mathbf{e}}^r, \overline{Z}^r)$ converges in distribution as $r \downarrow 0$ towards a limiting pair $(\overline{\mathbf{e}}^0, \overline{Z}^0)$, which is our conditioned Brownian snake with initial point 0. According to Theorem 2.2 below, the pair $(\overline{\mathbf{e}}^0, \overline{Z}^0)$ is the scaling limit of the pair consisting of the contour function and the spatial contour function of our spatial trees conditioned to have nonnegative spatial positions.

The preceding description of the conditioned process $(\overline{\mathbf{e}}^0, \overline{Z}^0)$ as the limit of $(\overline{\mathbf{e}}^r, \overline{Z}^r)$ when $r \downarrow 0$ does not give much information about this process. Note in particular that the underlying conditioning is in a sense very degenerate, since we are dealing with a continuous tree of linear Brownian paths all started from the origin and conditioned not to hit the negative half-line. Still Theorem 1.2 in [24] provides a useful construction of the conditioned object $(\overline{\mathbf{e}}^0, \overline{Z}^0)$ from the unconditioned one $(\mathbf{e}, Z^0)$. In order to present this construction, first recall that there is a.s. a unique s_* in $(0, 1)$ such that

$$Z^0(s_*) = \inf_{s\in[0,1]} Z^0(s)$$

(see Lemma 16 in [27] or Proposition 2.5 in [24]). For every $s \in [0, \infty)$, write $\{s\}$ for the fractional part of s. According to Theorem 1.2 in [24], the conditioned snake $(\overline{\mathbf{e}}^0, \overline{Z}^0)$ may be constructed explicitly as follows. For every $s \in [0, 1]$,

$$\overline{\mathbf{e}}^0(s) = \mathbf{e}(s_*) + \mathbf{e}(\{s_* + s\}) - 2 \inf_{s\wedge\{s_*+s\}\leq u\leq s\vee\{s_*+s\}} \mathbf{e}(u)$$

$$\overline{Z}^0(s) = Z^0(\{s_* + s\}) - Z^0(s_*).$$

In terms of trees, this means that the conditioned tree is obtained by re-rooting the unconditioned one at the vertex having the minimal spatial position. The above formula for $\overline{\mathbf{e}}^0(s)$ corresponds to the contour function for the tree coded by $\mathbf{e}$ re-rooted at the vertex s_* — see the discussion in the introduction of [24]. This construction of the conditioned snake $(\overline{\mathbf{e}}^0, \overline{Z}^0)$ is of course reminiscent of a famous result of Vervaat [32] connecting the normalized Brownian excursion and the Brownian bridge.

The initial motivation for the present work came from a recent paper of Chassaing and Schaeffer [8] discussing asymptotics for rooted planar maps (see also [7] for related results). A key result (Theorem 1 in [8] or Theorem 8.1 below) establishes a bijection between rooted quadrangulations with n faces and the so-called well-labelled trees with n edges. In the terminology of the present work, a well-labelled tree is a spatial tree $(\mathcal{T}, U)$ with the additional properties that the spatial positions are positive integers, the spatial position of the root is 1 and the spatial positions of two neighboring vertices can differ by at most 1. The preceding bijection between quadrangulations and trees has the nice feature that distances of vertices of the quadrangulation from the root correspond to spatial positions in the associated tree. This suggests that asymptotic properties of distances from the root in random quadrangulations with n faces can be read from asymptotics for well-labelled trees with n edges, an idea which was exploited in [8]. As an application of Theorem 2.2, we provide a direct proof of some of the main results of [8]. The key idea is to observe that uniform well-labelled trees with n edges can be obtained as conditioned spatial trees generated by letting the offspring distribution be geometric with parameter $1/2$, and the spatial distribution be uniform on $\{-1, 0, 1\}$. It then follows from Theorem 2.2 that the scaling limit of well-labelled trees with n vertices is described by our conditioned Brownian snake. As a consequence, several quantities such as the (suitably rescaled) radius of the quadrangulation have a limit in distribution which can be expressed, first in terms of the conditioned Brownian snake and then via the Vervaat-like transformation in terms of the unconditioned snake.

Another recent paper [27] of Marckert and Mokkadem proposes a model called the Brownian map for the continuous limit of rooted quadrangulations with n faces. This construction makes heavy use of the conditioned Brownian snake, which is defined in [27] via the Vervaat transformation rather than as a conditioned object as here or in [24]. Theorem 2.2 readily gives a positive answer to a conjecture of [27] (cf. Remark 6 in [27]) asserting that the Brownian map is, in some sense, the scaling limit of uniform rooted quadrangulations with n faces.

Connections between trees and planar maps are also of interest in theoretical physics: see in particular [3–5] and [15] for discussions and various applications. In this perspective, quadrangulations, or more general planar maps, serve as a model of random geometry. We also mention the article [6], which extends the bijection between quadrangulations and well-labelled trees to more general classes of planar maps. In a very recent paper [25], the bijections of [6] are used in order to generalize the Chassaing–Schaeffer asymptotics to general planar maps. However, in contrast with [8] and Section 8 of the present work, the article [25] deals with planar maps that are both rooted and pointed. It is likely that our methods can be applied to prove analogues of certain results of [25] in the context of rooted planar maps.

Outline of the paper

The following outline should help the reader to understand the main steps of the proof of Theorem 2.2, which is quite involved.

Section 3 introduces the key technical idea of comparing the distribution of spatial trees re-rooted at the minimal spatial position with that of conditioned trees. In the continuous framework, the Vervaat transformation shows that the distribution of the re-rooted tree and that of the conditioned one are identical. This is no longer true in a discrete setting, but there are still close relations between the two distributions (Lemmas 3.3 and 3.4), which play a major role in our proofs. As a first application, Section 4 derives estimates for the probability that the spatial positions are all positive: this probability is bounded above and below by constants times n^{-1}, where n is the number of vertices in the tree (Proposition 4.2).

Then, Section 5 briefly discusses a spatial Markov property for our spatial trees, which holds for the subtrees originating from the "first" vertices whose spatial position exceeds a level $a > 0$. This spatial Markov property is classical for unconstrained spatial trees, and it is easily extended to our conditioned objects.

Section 6 derives some asymptotic regularity properties of conditioned trees with n vertices. In particular, Proposition 6.1 implies that for any $\delta > 0$, with a probability close to 1 uniformly in n, the rescaled spatial contour function of the conditioned tree will be bounded below by a positive constant c (independent of n), over the interval $[\delta, 1-\delta]$. It follows that for most of the vertices in the conditioned tree, the associated spatial position is larger than $cn^{1/4}$. Such properties are first established for unconditioned trees via Theorem 2.1. They can then be transferred to conditioned trees thanks to Section 3.

Section 7 contains the proof of Theorem 2.2. Roughly speaking, the argument goes as follows. Thanks to the regularity properties of Section 6 and the spatial Markov property of Section 5, the conditioned tree is well approximated by a spatial tree (with a number of vertices of order n) whose root is located at a point close to $\alpha n^{1/4}$ (where α is a "small" positive number) and which is conditioned not to hit the negative half-line. The limit theorem for unconditioned spatial trees (Theorem 2.1) then shows that the limit of this suitably rescaled spatial tree is described by the Brownian snake $(\overline{\mathbf{e}}^{\alpha}, \overline{Z}^{\alpha})$ with initial point α and conditioned not to hit the negative half-line. Notice that the conditioning is not degenerate here since $\alpha > 0$. To complete the proof, we just have to use the fact that $(\overline{\mathbf{e}}^{\alpha}, \overline{Z}^{\alpha})$ is close in distribution to $(\overline{\mathbf{e}}^{0}, \overline{Z}^{0})$ when α is small, as was mentioned above.

Finally, Section 8 discusses applications to random quadrangulations.

2. Basic assumptions and statement of the main result

Throughout this work, we denote by μ the underlying offspring distribution, which is a probability measure on $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. We always assume that $\mu(1) < 1$ and

- μ is critical, that is $\sum_{k=0}^{\infty} k\mu(k) = 1$.
- μ is aperiodic, that is μ is not supported on a proper subgroup of $\mathbb{Z}$.
- μ has exponential moments: there exists a constant $\lambda > 0$ such that

$$\sum_{k=0}^{\infty} \mu(k)\, e^{\lambda k} < \infty.$$

We denote by $\sigma^2 > 0$ the variance of μ.

The law of the spatial displacement is denoted by γ. Thus γ is a probability distribution on $\mathbb{R}$. We exclude the trivial case $\gamma = \delta_0$ and we always assume that γ is symmetric, that is γ is invariant under the transformation $x \longrightarrow -x$. Our main results also require the additional assumption

$$\lim_{x\to\infty} x^4\, \gamma([x, \infty)) = 0. \tag{1}$$

When (1) holds, we denote by $\rho^2 > 0$ the variance of γ.

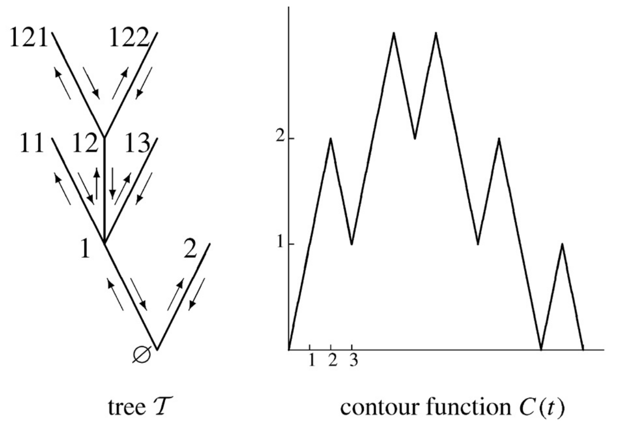

Fig. 1. A tree and its contour function.

Let us now introduce some formalism for discrete trees, which we borrow from Neveu [28]. Let

$$\mathcal{U} = \bigcup_{n=0}^{\infty} \mathbb{N}^n$$

where $\mathbb{N} = \{1, 2, \ldots\}$ and by convention $\mathbb{N}^0 = \{\varnothing\}$. If $u = (u_1, \ldots, u_m)$ and $v = (v_1, \ldots, v_n)$ belong to $\mathcal{U}$, we write $uv = (u_1, \ldots, u_m, v_1, \ldots, v_n)$ for the concatenation of u and v. In particular $u\varnothing = \varnothing u = u$. If $p \geq 1$ is an integer, we will write 1^p for the p-tuple $(1, 1, \ldots, 1) \in \mathbb{N}^p$. Finally, we set $\mathcal{U}^* = \mathcal{U} \setminus \{\varnothing\}$.

A plane tree $\mathcal{T}$ is a finite subset of $\mathcal{U}$ such that:

(i) $\varnothing \in \mathcal{T}$.
(ii) If $v \in \mathcal{T}$ and $v = uj$ for some $u \in \mathcal{U}$ and $j \in \mathbb{N}$, then $u \in \mathcal{T}$.
(iii) For every $u \in \mathcal{T}$, there exists a number $N_u(\mathcal{T}) \geq 0$ such that, for every $j \in \mathbb{N}$, $uj \in \mathcal{T}$ if and only if $1 \leq j \leq N_u(\mathcal{T})$.

We denote by $\mathbf{T}$ the set of all plane trees.

To define now the *contour function* of $\mathcal{T}$, consider a particle which starting from the root traverses the tree along its edges at speed one. It moves towards that vertex which is the smallest in lexicographical ordering among all vertices not visited so far, and from the last vertex moves back towards the root. Since each edge will be crossed twice, the total time needed to explore the tree is $2(|\mathcal{T}| - 1)$, where $|\mathcal{T}|$ denotes the cardinality (number of vertices) of $\mathcal{T}$. For every $t \in [0, 2(|\mathcal{T}| - 1)]$, we let $C(t)$ denote the distance from the root of the position of the particle at time t. Fig. 1 explains the definition of the contour function better than a formal definition. Clearly a tree $\mathcal{T}$ is uniquely determined by its contour function.

A (discrete) spatial tree is a pair $(\mathcal{T}, U)$, where $\mathcal{T} \in \mathbf{T}$ and $U = (U_v, v \in \mathcal{T})$ is a mapping from the set $\mathcal{T}$ into $\mathbb{R}$. We denote by Ω the set of all spatial trees. A spatial tree $(\mathcal{T}, U)$ can be coded by a pair (C, V), where $C = (C(t), 0 \leq t \leq 2(|\mathcal{T}| - 1))$ is the contour function of $\mathcal{T}$ and the *spatial contour function* $V = (V(t), 0 \leq t \leq 2(|\mathcal{T}| - 1))$ is defined as follows. First if t is an integer, then t corresponds in the evolution of the contour to a vertex v of $\mathcal{T}$, and we put $V(t) = U_v$. We then complete the definition of V by interpolating linearly between successive

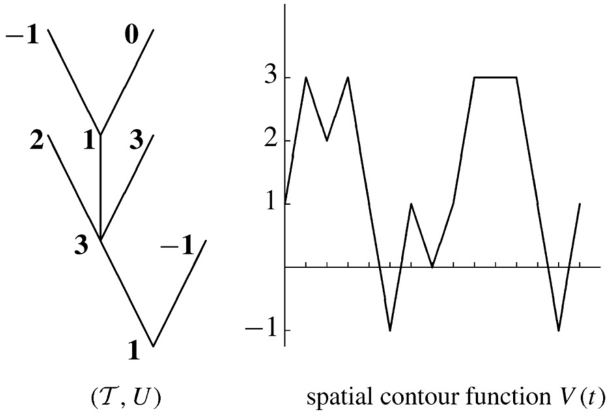

Fig. 2. A spatial tree and its spatial contour function.

integers. See Fig. 2 for an example. The tree on the left side of this figure is the same as in Fig. 1, but the numbers in bold attached to the different vertices now represent the spatial positions.

We denote by $\Pi(\mathrm{d}\mathcal{T})$ the law of the Galton–Watson tree with offspring distribution μ, which is a probability measure on $\mathbf{T}$. If $\mathcal{T} \in \mathbf{T}$ and $v_0 \in \mathcal{T}$, let

$$\mathcal{T}^{[v_0]} := \{v \in \mathcal{U} : v_0 v \in \mathcal{T}\}$$

denote the subtree of $\mathcal{T}$ originating from v_0. The probability measure Π is characterized by the following two properties [28]:

(i) $\Pi(N_\varnothing = j) = \mu(j)$ for every $j \geq 0$.

(ii) Under the conditional measure $\Pi(\cdot \mid N_\varnothing = j)$, the subtrees $\mathcal{T}^{[1]}, \mathcal{T}^{[2]}, \ldots, \mathcal{T}^{[j]}$ are independent and distributed according to Π.

The probability measure $\mathbb{P}_x(\mathrm{d}\mathcal{T}\mathrm{d}U)$ on the space Ω is then defined by

$$\mathbb{P}_x(\mathrm{d}\mathcal{T}\mathrm{d}U) = \Pi(\mathrm{d}\mathcal{T})\, R_x(\mathcal{T}, \mathrm{d}U)$$

where, for every $\mathcal{T} \in \mathbf{T}$, the probability measure $R_x(\mathcal{T}, \mathrm{d}U)$ is characterized as follows. Let $\mathcal{E}_\mathcal{T}$ denote the set of all edges of $\mathcal{T}$ and let $(Y_e, e \in \mathcal{E}_\mathcal{T})$ be i.i.d. random variables with distribution γ. For every $v \in \mathcal{T}$, set

$$X_v = x + \sum_{e \in [\varnothing, v]} Y_e,$$

where the notation $e \in [\varnothing, v]$ means that the edge e belongs to the ancestral line of v. Then $R_x(\mathcal{T}, \mathrm{d}U)$ is the distribution of $(X_v, v \in \mathcal{T})$.

Let us recall a well-known formula for the distribution of $|\mathcal{T}|$ under Π (see e.g. Section 5.2 of [29]). Let $(S_n)_{n\geq 0}$ be a random walk on $\mathbb{Z}$ with jump distribution $\nu(k) = \mu(k+1)$, $k = -1, 0, 1, 2, \ldots$, started from the origin and defined under the probability measure P. Set $\tau := \inf\{n \geq 0 : S_n = -1\}$. Then, for every integer $n \geq 1$,

$$\Pi(|\mathcal{T}| = n) = P(\tau = n) = \frac{1}{n} P(S_n = -1).$$

The aperiodicity of μ now implies that the latter quantity is positive for every n sufficiently large, so that we can define

$$\Pi^n(\mathrm{d}\mathcal{T}) = \Pi(\mathrm{d}\mathcal{T} \mid |\mathcal{T}| = n+1)$$
$$\mathbb{P}^n_x(\mathrm{d}\mathcal{T}\mathrm{d}U) = \mathbb{P}_x(\mathrm{d}\mathcal{T}\mathrm{d}U \mid |\mathcal{T}| = n+1).$$

Later, each time we will consider the probability measures Π^n or $\mathbb{P}^n_x$, it will be implicit that n is large enough so that this definition makes sense.

The following result is a special case of Theorem 2 in [19]. Recall that $(\mathbf{e}, Z^0)$ denotes the (one-dimensional) Brownian snake with initial point 0, as defined in Section 1.

Theorem 2.1. *Assume that (1) holds. Then the law under $\mathbb{P}^n_0$ of*

$$\left(\left(\frac{\sigma}{2}\,\frac{C(2nt)}{n^{1/2}}\right)_{0\le t\le 1}, \left(\frac{1}{\rho}\left(\frac{\sigma}{2}\right)^{1/2}\frac{V(2nt)}{n^{1/4}}\right)_{0\le t\le 1}\right)$$

converges as $n \to \infty$ to the law of $(\mathbf{e}, Z^0)$. The convergence holds in the sense of weak convergence of probability measures on $C([0,1],\mathbb{R})^2$.

We aim at proving a conditional version of Theorem 2.1. If $(\mathcal{T}, U)$ is a spatial tree, we set

$$\underline{U} = \inf\{U_v : v \in \mathcal{T},\ v \neq \varnothing\}$$

with the convention that $\underline{U} = +\infty$ if $\mathcal{T} = \{\varnothing\}$. For every $x \ge 0$, we then define

$$\overline{\mathbb{P}}^n_x(\cdot) := \mathbb{P}^n_x(\cdot \mid \underline{U} > 0).$$

Recall from Section 1 the notation $(\overline{\mathbf{e}}^0, \overline{Z}^0)$ for the conditioned Brownian snake. In the notation of [24], the distribution of $(\overline{\mathbf{e}}^0, \overline{Z}^0)$ is the law of the pair $(\zeta, \widehat{W})$ under $\overline{\mathbb{N}}^{(1)}_0$.

We can now state our main result.

Theorem 2.2. *Assume that (1) holds and let $x \ge 0$. Then the law under $\overline{\mathbb{P}}^n_x$ of*

$$\left(\left(\frac{\sigma}{2}\,\frac{C(2nt)}{n^{1/2}}\right)_{0\le t\le 1}, \left(\frac{1}{\rho}\left(\frac{\sigma}{2}\right)^{1/2}\frac{V(2nt)}{n^{1/4}}\right)_{0\le t\le 1}\right)$$

converges as $n \to \infty$ to the law of $(\overline{\mathbf{e}}^0, \overline{Z}^0)$. The convergence holds in the sense of weak convergence of probability measures on $C([0,1],\mathbb{R})^2$.

Remark. A trivial translation argument shows that $\mathbb{P}^n_0$ in Theorem 2.1 could be replaced by $\mathbb{P}^n_x$ for any $x \in \mathbb{R}$. In the setting of Theorem 2.2 however, no obvious argument can be used to reduce the proof to one particular value of x.

3. Re-rooting spatial trees

Recall that

$$\mathcal{U} = \bigcup_{n=0}^{\infty} \mathbb{N}^n$$

denotes the set of all possible vertices in our discrete trees, and that $\mathcal{U}^* = \mathcal{U} \setminus \{\varnothing\}$.

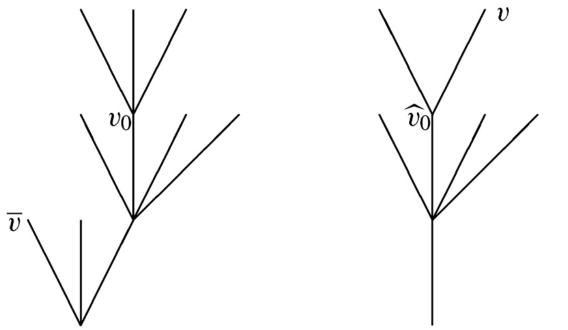

Fig. 3. A tree $\mathcal{T}$ and the re-rooted tree $\widehat{\mathcal{T}}^{(v_0)}$. If v is a vertex of the re-rooted tree, $\overline{v}$ denotes the corresponding vertex in the initial tree.

Let $v_0 \in \mathcal{U}^*$ and let $\mathcal{T} \in \mathbf{T}$ such that $v_0 \in \mathcal{T}$. Let $k = k(v_0, \mathcal{T})$ be the time of the first visit of v_0 in the evolution of the contour of $\mathcal{T}$. Also let $\ell = \ell(v_0, \mathcal{T})$ be the time of the last visit of v_0. Note that $\ell \geq k$ and $\ell = k$ iff v_0 is a leaf of $\mathcal{T}$. To simplify notation, we set $\zeta(\mathcal{T}) = 2(|\mathcal{T}| - 1)$.

For every $t \in [0, \zeta(\mathcal{T}) - (\ell - k)]$, set

$$\widehat{C}^{(v_0)}(t) = C(k) + C([\![k-t]\!]) - 2 \inf_{[\![k-t]\!] \wedge k \leq t \leq [\![k-t]\!] \vee k} C(r)$$

where $C(\cdot)$ is, as above, the contour function of $\mathcal{T}$, and $[\![k-t]\!]$ stands for the unique element of $[0, \zeta(\mathcal{T}))$ such that $[\![k-t]\!] - (k-t) = 0$ or $\zeta(\mathcal{T})$. We also set $\widehat{C}^{(v_0)}(t) = 0$ if $t > \zeta(\mathcal{T}) - (\ell - k)$.

Then it is easy to verify that there exists a unique plane tree $\widehat{\mathcal{T}}^{(v_0)} \in \mathbf{T}$ whose contour function is $\widehat{C}^{(v_0)}$. Informally, $\widehat{\mathcal{T}}^{(v_0)}$ is obtained by removing all vertices that are descendants of v_0 and then re-rooting the resulting tree at v_0 (we should also specify the ordering of children in the re-rooted tree, but we omit details in this informal description). See Fig. 3 for an example.

We note that $|\widehat{\mathcal{T}}^{(v_0)}| = |\mathcal{T}|$ iff v_0 is a leaf of $\mathcal{T}$. Also, if $v_0 = (j_1, j_2, \ldots, j_p)$, then $\widehat{v}_0 := (1, j_p, j_{p-1}, \ldots, j_2)$ automatically belongs to $\widehat{\mathcal{T}}^{(v_0)}$. Indeed, $\widehat{v}_0$ is the vertex of the re-rooted tree corresponding to the root of the initial tree. In Fig. 3, $v_0 = (3, 2)$ and $\widehat{v}_0 = (1, 2)$.

By definition, $\varnothing$ has exactly one child in the re-rooted tree $\widehat{\mathcal{T}}^{(v_0)}$. We define a new probability measure Q on $\mathbf{T}$ by setting

$$Q(\mathrm{d}\mathcal{T}) = \Pi(\mathrm{d}\mathcal{T} \mid N_\varnothing = 1),$$

where $N_\varnothing$ is the number of children of $\varnothing$ in $\mathcal{T}$. The conditioning a priori makes sense only if $\mu(1) > 0$. However, even when $\mu(1) = 0$, there is an obvious way of defining Q.

If $\mathcal{T} \in \mathbf{T}$ and $w_0 \in \mathcal{T}$, we also denote by $\mathcal{T}^{(w_0)}$ the new tree obtained from $\mathcal{T}$ by removing those vertices which are descendants of w_0 not equal to w_0.

Lemma 3.1. *Let $v_0 \in \mathcal{U}^*$ of the form $v_0 = (1, j_2, \ldots, j_p)$ for some $p \geq 1$, $j_2, \ldots, j_p \in \mathbb{N}$. Assume that $Q(v_0 \in \mathcal{T}) > 0$. Then the law under $Q(\cdot \mid v_0 \in \mathcal{T})$ of the re-rooted tree $\widehat{\mathcal{T}}^{(v_0)}$ coincides with the law under $Q(\cdot \mid \widehat{v}_0 \in \mathcal{T})$ of the tree $\mathcal{T}^{(\hat{v}_0)}$.*

The proof is an elementary application of properties of Galton–Watson trees. We leave details to the reader.

We shall need a spatial version of Lemma 3.1. Let $\mathbb{Q}$ be the probability measure on Ω defined by

$$\mathbb{Q}(\mathrm{d}\mathcal{T}\mathrm{d}U) = \mathbb{P}_0(\mathrm{d}\mathcal{T}\mathrm{d}U \mid N_\varnothing = 1) = Q(\mathrm{d}\mathcal{T})\, R_0(\mathcal{T}, \mathrm{d}U).$$

If $(\mathcal{T}, U) \in \Omega$ and $v_0 \in \mathcal{T}\backslash\{\varnothing\}$, the re-rooted spatial tree $(\widehat{\mathcal{T}}^{(v_0)}, \widehat{U}^{(v_0)})$ is defined as follows. For every vertex v of $\widehat{\mathcal{T}}^{(v_0)}$, $\widehat{U}^{(v_0)}_v = U_{\overline{v}} - U_{v_0}$, if $\overline{v}$ is the vertex of the initial tree $\mathcal{T}$ corresponding to v in $\widehat{\mathcal{T}}^{(v_0)}$ (see Fig. 3 for an example). Alternatively, we may say that the spatial contour function $\widehat{V}^{(v_0)}$ of $(\widehat{\mathcal{T}}^{(v_0)}, \widehat{U}^{(v_0)})$ is determined by

$$\widehat{V}^{(v_0)}(t) = V([\![k - t]\!]) - V(k)$$

for $0 \le t \le \zeta(\widehat{\mathcal{T}}^{(v_0)}) = \zeta(\mathcal{T}) - (\ell - k)$. (Here $k = k(v_0, \mathcal{T})$ and $\ell = \ell(v_0, \mathcal{T})$ are as in the beginning of the section.)

Lemma 3.2. *Let $v_0 \in \mathcal{U}^*$ be of the form $v_0 = (1, j_2, \ldots, j_p)$ for some $p \ge 1$, $j_2, \ldots, j_p \in \mathbb{N}$. Assume that $Q(v_0 \in \mathcal{T}) > 0$. Then the law under $\mathbb{Q}(\cdot \mid v_0 \in \mathcal{T})$ of the re-rooted tree $(\widehat{\mathcal{T}}^{(v_0)}, \widehat{U}^{(v_0)})$ coincides with the law under $\mathbb{Q}(\cdot \mid \widehat{v}_0 \in \mathcal{T})$ of the spatial tree $(\mathcal{T}^{(\hat{v}_0)}, U^{(\hat{v}_0)})$, where $U^{(\hat{v}_0)}$ denotes the restriction of U to $\mathcal{T}^{(\hat{v}_0)}$.*

Lemma 3.2 is a simple consequence of Lemma 3.1 and our definitions. Note that we use the symmetry of the spatial distribution γ.

If $(\mathcal{T}, U)$ is a spatial tree, we denote by $\Delta = \Delta(\mathcal{T}, U)$ the set of all vertices with minimal spatial position:

$$\Delta = \left\{ v \in \mathcal{T} : U_v = \min_{w \in \mathcal{T}} U_w \right\}.$$

We also denote by v_m the first element of Δ in lexicographical order. Finally, we use the notation ∂T for the set of all leaves of $\mathcal{T}$.

Lemma 3.3. *For any nonnegative measurable functional F on Ω,*

$$\mathbb{Q}\left(F(\widehat{\mathcal{T}}^{(v_m)}, \widehat{U}^{(v_m)}) \mathbf{1}_{\{|\Delta|=1, v_m \in \partial \mathcal{T}\}} \right) = \mathbb{Q}\left(F(\mathcal{T}, U)\, |\partial \mathcal{T}|\, \mathbf{1}_{\{\underline{U} > 0\}} \right).$$

This lemma provides a precise connection between the spatial tree re-rooted at the (first) vertex with minimal spatial position, and the initial tree conditioned to have positive spatial positions. Compare with Theorem 1.2 in [24], but notice the multiplicative factor $|\partial\mathcal{T}|$ that occurs in the discrete setting.

Proof. Let $v_0 \in \mathcal{U}^*$ such that $Q(v_0 \in \mathcal{T}) > 0$. Then

$$\mathbb{Q}\left(F(\widehat{\mathcal{T}}^{(v_0)}, \widehat{U}^{(v_0)}) \mathbf{1}_{\{\Delta = \{v_0\}\}} \right) = \mathbb{Q}\left(F(\widehat{\mathcal{T}}^{(v_0)}, \widehat{U}^{(v_0)}) \mathbf{1}_{\{v_0 \in \mathcal{T};\, U_v > U_{v_0},\, \forall v \in \mathcal{T}\backslash\{v_0\}\}} \right). \tag{2}$$

Recall from Section 2 the notation $\mathcal{T}^{[v_0]} := \{v \in \mathcal{U} : v_0 v \in \mathcal{T}\}$ for the subtree of $\mathcal{T}$ originating from v_0. For every $v \in \mathcal{T}^{[v_0]}$, put

$$U^{[v_0]}_v := U_{v_0 v} - U_{v_0}.$$

Plainly, under the probability measure $\mathbb{Q}(\cdot \mid v_0 \in \mathcal{T})$, the spatial tree $(\mathcal{T}^{[v_0]}, U^{[v_0]})$ is independent of $(\mathcal{T}^{(v_0)}, U^{(v_0)})$ and has distribution $\mathbb{P}_0$. Since $(\widehat{\mathcal{T}}^{(v_0)}, \widehat{U}^{(v_0)})$ is by construction a function of $(\mathcal{T}^{(v_0)}, U^{(v_0)})$, we can rewrite formula (2) as follows:

$$\mathbb{Q}\left(F(\widehat{\mathcal{T}}^{(v_0)}, \widehat{U}^{(v_0)})\mathbf{1}_{\{\Delta=\{v_0\}\}}\right) = \mathbb{Q}\left(F(\widehat{\mathcal{T}}^{(v_0)}, \widehat{U}^{(v_0)})\mathbf{1}_{\{v_0\in\mathcal{T};\, U_v>U_{v_0},\forall v\in\mathcal{T}^{(v_0)}\setminus\{v_0\}\}}\right) \times \mathbb{P}_0(\underline{U}>0). \tag{3}$$

Using Lemma 3.2, we then get

$$\begin{aligned}
&\mathbb{Q}\left(F(\widehat{\mathcal{T}}^{(v_0)}, \widehat{U}^{(v_0)})\mathbf{1}_{\{v_0\in\mathcal{T};\, U_v>U_{v_0},\forall v\in\mathcal{T}^{(v_0)}\setminus\{v_0\}\}}\right)\\
&\quad= \mathbb{Q}\left(F(\widehat{\mathcal{T}}^{(v_0)}, \widehat{U}^{(v_0)})\mathbf{1}_{\{v_0\in\mathcal{T};\, \hat{U}^{(v_0)}_v>0,\, \forall v\in\hat{\mathcal{T}}^{(v_0)}\setminus\{\varnothing\}\}}\right)\\
&\quad= \mathbb{Q}\left(F(\mathcal{T}^{(\hat{v}_0)}, U^{(\hat{v}_0)})\mathbf{1}_{\{\hat{v}_0\in\mathcal{T};\, U^{(\hat{v}_0)}_v>0,\, \forall v\in\mathcal{T}^{(\hat{v}_0)}\setminus\{\varnothing\}\}}\right).
\end{aligned}$$

Now notice that $(\mathcal{T}^{(v)}, U^{(v)}) = (\mathcal{T}, U)$ if $v \in \partial\mathcal{T}$. Moreover, the event $\{v \in \partial\mathcal{T}\}$ is independent of $(\mathcal{T}^{(v)}, U^{(v)})$ under $\mathbb{Q}(\cdot \mid v \in \mathcal{T})$, and has probability $\mu(0)$. Combining these observations, we get

$$\begin{aligned}
&\mathbb{Q}\left(F(\widehat{\mathcal{T}}^{(v_0)}, \widehat{U}^{(v_0)})\mathbf{1}_{\{v_0\in\mathcal{T};\, U_v>U_{v_0},\forall v\in\mathcal{T}^{(v_0)}\setminus\{v_0\}\}}\right)\\
&\quad= \frac{1}{\mu(0)}\,\mathbb{Q}\left(F(\mathcal{T}, U)\mathbf{1}_{\{\hat{v}_0\in\partial\mathcal{T};\, U_v>0,\, \forall v\in\mathcal{T}\setminus\{\varnothing\}\}}\right).
\end{aligned}$$

Using (3) and the preceding equalities, we get

$$\mathbb{Q}\left(F(\widehat{\mathcal{T}}^{(v_0)}, \widehat{U}^{(v_0)})\mathbf{1}_{\{\Delta=\{v_0\}\}}\right) = \frac{\mathbb{P}_0(\underline{U}>0)}{\mu(0)}\,\mathbb{Q}\left(F(\mathcal{T}, U)\mathbf{1}_{\{\hat{v}_0\in\partial\mathcal{T};\, U_v>0,\, \forall v\in\mathcal{T}\setminus\{\varnothing\}\}}\right). \tag{4}$$

From the property stated just before (3), we easily see that under the conditional measure $\mathbb{Q}(\cdot \mid \Delta = \{v_0\})$, the spatial tree $(\mathcal{T}^{[v_0]}, U^{[v_0]})$ is independent of $(\mathcal{T}^{(v_0)}, U^{(v_0)})$ and has distribution $\mathbb{P}_0(\cdot \mid \underline{U} > 0)$. Hence,

$$\mathbb{Q}\left(F(\widehat{\mathcal{T}}^{(v_0)}, \widehat{U}^{(v_0)})\mathbf{1}_{\{\Delta=\{v_0\}\}}\right) = \frac{1}{\mathbb{P}_0(\mathcal{T}=\{\varnothing\}\mid \underline{U}>0)} \times \mathbb{Q}\left(F(\widehat{\mathcal{T}}^{(v_0)}, \widehat{U}^{(v_0)})\mathbf{1}_{\{v_0\in\partial\mathcal{T},\, \Delta=\{v_0\}\}}\right). \tag{5}$$

Since

$$\mathbb{P}_0(\mathcal{T}=\{\varnothing\}\mid \underline{U}>0) = \frac{\mu(0)}{\mathbb{P}_0(\underline{U}>0)},$$

(4) and (5) give

$$\mathbb{Q}\left(F(\widehat{\mathcal{T}}^{(v_0)}, \widehat{U}^{(v_0)})\mathbf{1}_{\{v_0\in\partial\mathcal{T},\, \Delta=\{v_0\}\}}\right) = \mathbb{Q}\left(F(\mathcal{T}, U)\mathbf{1}_{\{\hat{v}_0\in\partial\mathcal{T},\, \underline{U}>0\}}\right). \tag{6}$$

Summing (6) over all possible choices of v_0 leads to the desired result. □

We shall need a variant of Lemma 3.3.

Lemma 3.4. *For any nonnegative measurable functional F on Ω,*

$$\mathbb{Q}\left(\sum_{v_0\in\Delta\cap\partial\mathcal{T}} F(\widehat{\mathcal{T}}^{(v_0)}, \widehat{U}^{(v_0)})\right) = \mathbb{Q}\left(F(\mathcal{T}, U)\,|\partial\mathcal{T}|\,\mathbf{1}_{\{\underline{U}\geq 0\}}\right).$$

Proof. By arguing as in the proof of (4), we get for every $v_0 \in \mathcal{U}^*$,

$$\begin{aligned}
\mathbb{Q}\left(F(\widehat{\mathcal{T}}^{(v_0)}, \widehat{U}^{(v_0)})\mathbf{1}_{\{v_0\in\Delta\}}\right) &= \mathbb{Q}\left(F(\widehat{\mathcal{T}}^{(v_0)}, \widehat{U}^{(v_0)})\mathbf{1}_{\{v_0\in\mathcal{T};\, U_v\geq U_{v_0},\, \forall v\in\mathcal{T}^{(v_0)}\}}\right) \\
&\quad \times \mathbb{P}_0(\underline{U}\geq 0) \\
&= \mathbb{Q}\left(F(\mathcal{T}^{(\hat{v}_0)}, U^{(\hat{v}_0)})\mathbf{1}_{\{\hat{v}_0\in\mathcal{T};\, U_v^{(\hat{v}_0)}\geq 0,\, \forall v\in\mathcal{T}^{(\hat{v}_0)}\}}\right) \\
&\quad \times \mathbb{P}_0(\underline{U}\geq 0) \\
&= \frac{\mathbb{P}_0(\underline{U}\geq 0)}{\mu(0)}\, \mathbb{Q}\left(F(\mathcal{T}, U)\mathbf{1}_{\{\hat{v}_0\in\partial\mathcal{T};\, U_v\geq 0,\, \forall v\in\mathcal{T}\}}\right).
\end{aligned}$$

On the other hand, analogously to (5),

$$\mathbb{Q}\left(F(\widehat{\mathcal{T}}^{(v_0)}, \widehat{U}^{(v_0)})\mathbf{1}_{\{v_0\in\Delta\}}\right) = \frac{1}{\mathbb{P}_0(\mathcal{T}=\{\varnothing\}\mid \underline{U}\geq 0)}\, Q\left(F(\widehat{\mathcal{T}}^{(v_0)}, \widehat{U}^{(v_0)})\,\mathbf{1}_{\{v_0\in\Delta\cap\partial\mathcal{T}\}}\right).$$

The lemma follows by combining the previous two identities and then summing over all choices of $v_0 \in \mathcal{U}^*$. $\square$

Remark. If γ has no atoms we have $|\Delta| = 1$, $\mathbb{Q}$ a.s., and Lemmas 3.3 and 3.4 reduce to the same identity. In our applications, we shall be concerned with the case when γ does have atoms, and is even supported on a finite subset of $\mathbb{Z}$.

4. Estimates for the probability of staying on the positive side

Our goal in this section is to derive upper and lower bounds for the probability $\mathbb{P}_x^n(\underline{U} > 0)$ when $n \to \infty$. Our main tools will be Lemmas 3.3 and 3.4. We also need a preliminary estimate concerning the cardinality $|\partial\mathcal{T}|$ of the set of leaves.

Lemma 4.1. *There exists a constant $\alpha_0 > 0$ such that, for every n sufficiently large,*

$$\mathbb{Q}(||\partial\mathcal{T}| - \mu(0)n| > n^{3/4}, |\mathcal{T}| = n+1) \leq e^{-n^{\alpha_0}}.$$

Proof. We first recall some classical facts about relations between random walks and Galton–Watson trees. As in Section 2 above, let $S_n = X_1 + \cdots + X_n$ be a random walk on $\mathbb{Z}$ with jump distribution $\nu(k) = \mu(k+1)$, $k = -1, 0, 1, 2, \ldots$, started from the origin and defined under the probability measure P. Set $\tau := \inf\{n \geq 0 : S_n = -1\}$, and for every integer $n \geq 1$,

$$M_n = |\{k \in \{1, 2, \ldots, n\} : X_k = -1\}|.$$

Then the law of the pair $(|\mathcal{T}|, |\partial\mathcal{T}|)$ under Q coincides with that of the pair $(1+\tau, M_\tau)$ under P. For a proof, see e.g. the discussion in Section 2 of [23], or Section 5.2 of [29].

It follows that

$$\begin{aligned}
\mathbb{Q}(||\partial\mathcal{T}| - \mu(0)n| > n^{3/4}, |\mathcal{T}| = n+1) &= P(|M_\tau - \mu(0)n| > n^{3/4}, \tau = n) \\
&\leq P(|M_n - \mu(0)n| > n^{3/4}).
\end{aligned}$$

The estimate of the lemma now follows from standard moderate deviation estimates for sums of independent Bernoulli variables. $\square$

We set $\mathbb{Q}^n(\cdot) = \mathbb{Q}(\cdot \mid |\mathcal{T}| = n+1)$, which makes sense for every n sufficiently large.

Proposition 4.2. *Let $K > 0$. There exist positive constants $c_1, c_2 = c_2(K), \widetilde{c}_1, \widetilde{c}_2$ such that, for every $x \in [0, K]$ and every n sufficiently large,*

$$\frac{\widetilde{c}_1}{n} \le \mathbb{Q}^n(\underline{U} > 0) \le \frac{\widetilde{c}_2}{n},$$
$$\frac{c_1}{n} \le \mathbb{P}^n_x(\underline{U} > 0) \le \frac{c_2}{n}.$$

Remark. By an obvious comparison argument, c_1 can be chosen independently of K.

Proof. We first bound $\mathbb{Q}^n(\underline{U} > 0)$. We apply Lemma 3.3 with

$$F(\mathcal{T}, U) = \mathbf{1}_{\{|\mathcal{T}|=n+1\}},$$

noting that $|\widehat{\mathcal{T}}^{(v)}| = |\mathcal{T}|$ if $v \in \partial\mathcal{T}$. It follows that

$$\mathbb{Q}(|\partial\mathcal{T}|\,\mathbf{1}_{\{|\mathcal{T}|=n+1,\underline{U}>0\}}) \le \mathbb{Q}(|\mathcal{T}| = n+1).$$

On the other hand, Lemma 4.1 shows that, for n sufficiently large,

$$\mathbb{Q}(|\partial\mathcal{T}|\,\mathbf{1}_{\{|\mathcal{T}|=n+1,\,\underline{U}>0\}}) \ge (\mu(0)n - n^{3/4})\,\mathbb{Q}(|\mathcal{T}| = n+1, \underline{U} > 0) - n\,e^{-n^{\alpha_0}}.$$

By combining this with the preceding bound, we get

$$\mathbb{Q}^n(\underline{U} > 0) \le \frac{1}{\mu(0)n - n^{3/4}} + \frac{n\,e^{-n^{\alpha_0}}}{(\mu(0)n - n^{3/4})\mathbb{Q}(|\mathcal{T}| = n+1)}. \tag{7}$$

With the notation of the proof of Lemma 4.1, we have also

$$\mathbb{Q}(|\mathcal{T}| = n+1) = P(\tau = n) = \frac{1}{n}\,P(S_n = -1) \underset{n\to\infty}{\sim} c_0\,n^{-3/2} \tag{8}$$

where c_0 is a positive constant, and the last estimate follows from a standard local limit theorem. We thus deduce from (7) that

$$\limsup_{n\to\infty} n\,\mathbb{Q}^n(\underline{U} > 0) \le \frac{1}{\mu(0)}, \tag{9}$$

yielding the desired upper bound for $\mathbb{Q}^n(\underline{U} > 0)$.

Let us now discuss a lower bound for $\mathbb{Q}^n(\underline{U} \ge 0)$. Applying Lemma 3.4 with the same function F, we get

$$\mathbb{Q}(|\partial\mathcal{T}|\,\mathbf{1}_{\{|\mathcal{T}|=n+1,\,\underline{U}\ge 0\}}) = \mathbb{Q}(|\Delta \cap \partial\mathcal{T}|\,\mathbf{1}_{\{|\mathcal{T}|=n+1\}}) \ge \mathbb{Q}(|\Delta \cap \partial\mathcal{T}| \ge 1,\ |\mathcal{T}| = n+1), \tag{10}$$

and we now need to show that the latter quantity is bounded below by $c\,\mathbb{Q}(|\mathcal{T}| = n+1)$ for some positive constant c. To this end, we state another lemma. Recall that v_m is the first vertex in Δ for the lexicographical order of vertices.

Lemma 4.3. *There exists a constant $\bar{c}_1 > 0$ such that, for every n sufficiently large,*

$$\mathbb{Q}^n(v_m \in \partial\mathcal{T}) \ge \bar{c}_1.$$

We postpone the proof of Lemma 4.3 and complete that of Proposition 4.2. As a consequence of Lemma 4.3, we have

$$\mathbb{Q}(|\Delta \cap \partial\mathcal{T}| \ge 1,\ |\mathcal{T}| = n+1) \ge \bar{c}_1\,\mathbb{Q}(|\mathcal{T}| = n+1).$$

Using now (10), we get

$$n\,\mathbb{Q}(|\mathcal{T}| = n+1,\ \underline{U} \geq 0) \geq \overline{c}_1\,\mathbb{Q}(|\mathcal{T}| = n+1),$$

and so

$$\mathbb{Q}^n(\underline{U} \geq 0) \geq \frac{\overline{c}_1}{n}. \tag{11}$$

The bounds involving $\mathbb{P}^n_x(\underline{U} > 0)$ are easily derived from (9) and (11). Consider first the lower bound. Clearly, it is enough to take $x = 0$. Then fix $k \geq 1$ with $\mu(k) > 0$. By arguing on an appropriate event contained in $\{N_\varnothing = k, N_1 = k\}$, we get

$$\mathbb{P}_0(\underline{U} > 0,\ |\mathcal{T}| = n+1) \geq \mu(k)^2\,\gamma((0,\infty))^{2k-1}\,\mu(0)^{2(k-1)} \\ \times \mathbb{Q}(\underline{U} \geq 0, |\mathcal{T}| = n+2-2k).$$

Since $\mathbb{P}_0(|\mathcal{T}| = n+1) \sim c_0\,n^{-3/2} \sim \mathbb{Q}(|\mathcal{T}| = n+2-2k)$ as $n \to \infty$, we see that the lower bound for $\mathbb{P}^n_0(\underline{U} > 0)$ follows from (11).

Consider then the upper bound. It is enough to take $x = K$. Let k be as above, then choose $y > 0$ such that $\gamma([y,\infty)) > 0$ and let $p \geq 1$ be an integer such that $py \geq K$. Again by arguing on an appropriate event contained in $\{N_1 = k, N_{(1,1)} = k, \ldots, N_{1^{p-1}} = k\}$, we get

$$\mathbb{Q}(\underline{U} > 0,\ |\mathcal{T}| = n+2+(p-1)k) \\ \geq \mu(k)^{p-1}\gamma([y,\infty))^p\mu(0)^{(p-1)(k-1)}\gamma((0,\infty))^{(p-1)(k-1)}\mathbb{P}_K(\underline{U} > 0, |\mathcal{T}| = n+1)$$

and thus the upper bound for $\mathbb{P}^n_K(\underline{U} > 0)$ follows from (9).

Finally, the bound $\mathbb{P}^n_0(\underline{U} > 0) \geq c_1/n$ readily implies $\mathbb{Q}^n(\underline{U} > 0) \geq \widetilde{c}_1/n$ with $\widetilde{c}_1 = \gamma((0,\infty))\,c_1$. □

Proof of Lemma 4.3. We first observe that under the probability measure $\mathbb{Q}(\cdot \mid v_m \neq \varnothing)$, the spatial tree $(\mathcal{T}^{[v_m]}, U^{[v_m]})$ is independent of $(\mathcal{T}^{(v_m)}, U^{(v_m)})$ and has distribution $\mathbb{P}_0(\cdot \mid \underline{U} \geq 0)$. Indeed, if we condition on the value of v_m, we see that this statement follows from basic properties of Galton–Watson trees, similar to those that were used in the proof of Lemma 3.3. We then get

$$\begin{aligned}\mathbb{Q}(|\mathcal{T}| = n+1,\ v_m \neq \varnothing) &= \sum_{k=0}^{n-1} \mathbb{Q}(|\mathcal{T}^{(v_m)}| = n-k+1,\ |\mathcal{T}^{[v_m]}| = k+1,\ v_m \neq \varnothing) \\ &= \sum_{k=0}^{n-1} \mathbb{Q}(|\mathcal{T}^{(v_m)}| = n-k+1,\ v_m \neq \varnothing)\,\eta(k+1) \end{aligned} \tag{12}$$

where $\eta(j) = \mathbb{P}_0(|\mathcal{T}| = j \mid \underline{U} \geq 0)$, for every $j \geq 1$. On the other hand, using once again the observation of the beginning of the proof, we have for every integer $\ell \geq 2$,

$$\begin{aligned}\mathbb{Q}(|\mathcal{T}| = \ell,\ v_m \in \partial\mathcal{T}) &= \mathbb{Q}(|\mathcal{T}^{(v_m)}| = \ell,\ \mathcal{T}^{[v_m]} = \{\varnothing\},\ v_m \neq \varnothing) \\ &= \eta(0)\,\mathbb{Q}(|\mathcal{T}^{(v_m)}| = \ell,\ v_m \neq \varnothing). \end{aligned} \tag{13}$$

Fix an integer $p \geq 2$ such that $\mu(p) > 0$. Under the probability measure $\mathbb{Q}(\cdot \mid N_1 = p)$, we can consider the p trees $\mathcal{T}_1, \ldots, \mathcal{T}_p$ defined as follows. For every $1 \leq j \leq p$, $\mathcal{T}_j$ consists of the root $\varnothing$ and of the vertices of the type $1v$ such that $1jv \in \mathcal{T}$. Then, under $\mathbb{Q}(\cdot \mid N_1 = p)$, $\mathcal{T}_1, \ldots, \mathcal{T}_p$ are independent and distributed according to Q. Let k be an integer with $p-1 \leq k \leq n-2$. We consider the event where $|\mathcal{T}_1| = n-k$, $|\mathcal{T}_2| = k-p+3$ and $|\mathcal{T}_j| = 2$ for other values of j.

By arguing on this event and imposing appropriate conditions on the spatial displacements (using in particular the fact that $\gamma((-\infty,0]) = \gamma([0,\infty)) \geq 1/2$), we get the bound

$$\begin{aligned}
&\mathbb{Q}(|\mathcal{T}| = n+1,\ v_m \in \partial\mathcal{T}) \\
&\geq \frac{\mu(p)}{2} \sum_{k=p-1}^{n-2} \mathbb{Q}(|\mathcal{T}| = n-k,\ v_m \in \partial\mathcal{T})\,\mathbb{Q}(|\mathcal{T}| = k-p+3,\ \underline{U} \geq 0)\left(\frac{\mu(0)}{2}\right)^{p-2} \\
&= c_{(p)} \sum_{k=p-1}^{n-2} \mathbb{Q}(|\mathcal{T}| = n-k,\ v_m \in \partial\mathcal{T})\,\mathbb{Q}(|\mathcal{T}| = k-p+3,\ \underline{U} \geq 0)
\end{aligned}$$

where $c_{(p)} > 0$ is a constant depending on p and μ.

Similarly, by requiring the spatial displacement along the first edge to be nonnegative, we get for $k \geq p-1$,

$$\begin{aligned}
\mathbb{Q}(|\mathcal{T}| = k-p+3,\ \underline{U} \geq 0) &\geq \frac{1}{2}\,\mathbb{P}_0(|\mathcal{T}| = k-p+2,\ \underline{U} \geq 0) \\
&= \frac{1}{2}\,\mathbb{P}_0(\underline{U} \geq 0)\,\eta(k-p+2).
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathbb{Q}(|\mathcal{T}| = n+1,\ v_m \in \partial\mathcal{T}) &\geq c'_{(p)} \sum_{k=p-1}^{n-2} \mathbb{Q}(|\mathcal{T}| = n-k,\ v_m \in \partial\mathcal{T})\,\eta(k-p+2) \\
&= c'_{(p)} \sum_{j=0}^{n-p-1} \mathbb{Q}(|\mathcal{T}| = (n-p)-j+1,\ v_m \in \partial\mathcal{T})\,\eta(j+1).
\end{aligned}$$

Using (13) we arrive at

$$\begin{aligned}
\mathbb{Q}(|\mathcal{T}| = n+1,\ v_m \in \partial\mathcal{T}) \geq c''_{(p)} \sum_{j=0}^{(n-p)-1} &\mathbb{Q}(|\mathcal{T}^{(v_m)}| = (n-p)-j+1,\ v_m \neq \varnothing) \\
&\times \eta(j+1). \qquad (14)
\end{aligned}$$

We compare the last bound with (12) written with n replaced by $n-p$. It follows that

$$\mathbb{Q}(|\mathcal{T}| = n+1,\ v_m \in \partial\mathcal{T}) \geq c''_{(p)}\,\mathbb{Q}(|\mathcal{T}| = n-p+1,\ v_m \neq \varnothing). \qquad (15)$$

Since $\mathbb{Q}(|\mathcal{T}| = n+1,\ v_m \neq \varnothing) \sim \mathbb{Q}(|\mathcal{T}| = n+1) \sim c_0 n^{-3/2}$ as $n \to \infty$, Lemma 4.3 follows from (15). $\square$

Remark. In view of Proposition 4.2, one expects the existence of a positive constant c_∞ such that

$$\lim_{n\to\infty} n\,\mathbb{Q}^n(\underline{U} > 0) = c_\infty.$$

When γ has no atoms, and under some additional conditions on the offspring distribution μ, this can indeed be proved from the lemmas of Section 3, with $c_\infty = \mathbb{P}_0(\underline{U} > 0)^{-1}$. Similar asymptotics for $\mathbb{P}_0^n(\underline{U} > 0)$ then follow easily. Since we do not need these more precise estimates, we will not address this problem here.

5. A spatial Markov property

In this section, we briefly discuss a Markov property for our branching trees, which will be used in the proof of our main result. Arguments are elementary and so we omit most details.

We fix $a > 0$. If $(\mathcal{T}, U)$ is a spatial tree and $v \in \mathcal{T}$, we say that v is an exit vertex from $(-\infty, a)$ if $U_v \geq a$ and $U_{v'} < a$ for every ancestor v' of v distinct from v. Denote by $v_1, v_2, \ldots, v_M$ the exit vertices from $(-\infty, a)$ listed in lexicographical order. If $i, j \in \{1, 2, \ldots, M\}$ and $i \neq j$, then v_i cannot be an ancestor of v_j.

For $v \in \mathcal{T}$, and for every $v' \in \mathcal{T}^{[v]}$, we set

$$\overline{U}^{[v]}_{v'} = U_{vv'}$$

(compare with the definition of $U^{[v]}$). Finally, we denote by $\mathcal{T}^a$ the subtree of $\mathcal{T}$ consisting of those vertices which are not strict descendants of $v_1, \ldots, v_M$. In particular, $v_1, \ldots, v_M \in \mathcal{T}^a$. We also denote by U^a the restriction of U to $\mathcal{T}^a$. Informally, $(\mathcal{T}^a, U^a)$ corresponds to the tree $(\mathcal{T}, U)$ "truncated at the first exit time" from $(-\infty, a)$.

Proposition 5.1. *Let $x \in [0, a)$ and $p \geq 1$. Under the probability measure $\mathbb{P}_x(\cdot \mid M = p)$, conditionally on $(\mathcal{T}^a, U^a)$, the spatial trees $(\mathcal{T}^{[v_1]}, \overline{U}^{[v_1]}), \ldots, (\mathcal{T}^{[v_p]}, \overline{U}^{[v_p]})$ are independent and distributed respectively according to $\mathbb{P}_{U_{v_1}}, \ldots, \mathbb{P}_{U_{v_p}}$.*

The proof of Proposition 5.1 is an easy application of the properties of Galton–Watson trees. We leave details to the reader. See e.g. [9] for closely related statements in a slightly different setting.

Conditional versions of Proposition 5.1 are derived in a straightforward way. Firstly, this statement remains valid if $\mathbb{P}_x$ is replaced by

$$\overline{\mathbb{P}}_x(\cdot) := \mathbb{P}_x(\cdot \mid \underline{U} > 0),$$

provided $\mathbb{P}_{U_{v_1}}, \ldots, \mathbb{P}_{U_{v_p}}$ in the conclusion are also replaced by $\overline{\mathbb{P}}_{U_{v_1}}, \ldots, \overline{\mathbb{P}}_{U_{v_p}}$.

Then, by conditioning with respect to the sizes of the various trees, we arrive at the following result.

Corollary 5.2. *Let $x \in [0, a)$ and $p \in \{1, \ldots, n\}$. Let $n_1, \ldots, n_p$ be positive integers such that $n_1 + \cdots + n_p \leq n$. Assume that*

$$\overline{\mathbb{P}}^n_x(M = p, |\mathcal{T}^{[v_1]}| = n_1, \ldots, |\mathcal{T}^{[v_p]}| = n_p) > 0.$$

Then, under the probability measure $\overline{\mathbb{P}}^n_x(\cdot \mid M = p, |\mathcal{T}^{[v_1]}| = n_1, \ldots, |\mathcal{T}^{[v_p]}| = n_p)$, conditionally on $(\mathcal{T}^a, U^a)$, the spatial trees $(\mathcal{T}^{[v_1]}, \overline{U}^{[v_1]}), \ldots, (\mathcal{T}^{[v_p]}, \overline{U}^{[v_p]})$ are independent and distributed respectively according to $\overline{\mathbb{P}}^{n_1}_{U_{v_1}}, \ldots, \overline{\mathbb{P}}^{n_p}_{U_{v_p}}$.

6. Asymptotic properties of conditioned trees

From now on we assume that (1) holds.

In view of our main result Theorem 2.2, it is convenient to introduce a specific notation for rescaled processes. For every integer $n \geq 1$ and every $t \in [0, 1]$, we set

$$C^{(n)}(t) = \frac{\sigma}{2} \, \frac{C(2nt)}{n^{1/2}},$$

$$V^{(n)}(t) = \frac{1}{\rho} \left(\frac{\sigma}{2}\right)^{1/2} \frac{V(2nt)}{n^{1/4}}.$$

Before proceeding to the proof of Theorem 2.2, we need to get some information about asymptotic properties of the pair $(C^{(n)}, V^{(n)})$ under $\overline{\mathbb{P}}_x^n$. We will consider the conditioned measure

$$\overline{\mathbb{Q}}^n := \mathbb{Q}^n(\cdot \mid \underline{U} > 0).$$

Proposition 6.1. *For every $b > 0$ and $\varepsilon \in (0, 1/10)$, we can find $\delta, \alpha \in (0, \varepsilon)$ such that, for all n sufficiently large,*

$$\overline{\mathbb{Q}}^n \left(\inf_{t\in[\delta/2,1-\delta/2]} V^{(n)}(t) \geq 2\alpha, \sup_{t\in[0,4\delta]\cup[1-4\delta,1]} (C^{(n)}(t) + V^{(n)}(t)) \leq \varepsilon/2 \right) \geq 1 - b.$$

Consequently, if $K > 0$, we have also for all n sufficiently large, for every $x \in [0, K]$,

$$\overline{\mathbb{P}}_x^n \left(\inf_{t\in[\delta,1-\delta]} V^{(n)}(t) \geq \alpha, \sup_{t\in[0,3\delta]\cup[1-3\delta,1]} (C^{(n)}(t) + V^{(n)}(t)) \leq \varepsilon \right) \geq 1 - c_3 b,$$

where the constant c_3 only depends on μ, γ and K.

The second part of the proposition will follow from the first one by arguments similar to those that were used in the proof of Proposition 4.2 above. To prove the first part of the proposition, we will use Theorem 2.1 together with the following crucial lemma.

Lemma 6.2. *Let F be a nonnegative measurable function on Ω such that $0 \leq F \leq 1$. There exist a finite constant $\overline{c}$, which does not depend on F nor on n, such that*

$$\overline{\mathbb{Q}}^n(F(\mathcal{T}, U)) \leq \overline{c}\, \mathbb{Q}^n(F(\widehat{\mathcal{T}}^{(v_m)}, \widehat{U}^{(v_m)})) + O(n^{5/2} e^{-n^{\alpha_0}}),$$

where α_0 is as in Lemma 4.1, and the estimate $O(n^{5/2} e^{-n^{\alpha_0}})$ for the remainder holds uniformly in F.

Proof. For every n sufficiently large,

$$\begin{aligned}
&\mathbb{Q}\left(F(\mathcal{T}, U)\, \mathbf{1}_{\{|\mathcal{T}|=n+1\}}\, \mathbf{1}_{\{\underline{U}>0\}}\right) \\
&\quad\leq \mathbb{Q}\left(F(\mathcal{T}, U)\, \mathbf{1}_{\{|\mathcal{T}|=n+1\}}\, \mathbf{1}_{\{\underline{U}>0\}} \frac{|\partial\mathcal{T}|}{\mu(0)n - n^{3/4}}\, \mathbf{1}_{\{|\partial\mathcal{T}|\geq\mu(0)n-n^{3/4}\}} \right) \\
&\qquad + \mathbb{Q}\left(F(\mathcal{T}, U)\, \mathbf{1}_{\{|\mathcal{T}|=n+1\}}\, \mathbf{1}_{\{\underline{U}>0\}}\, \mathbf{1}_{\{|\partial\mathcal{T}|<\mu(0)n-n^{3/4}\}}\right) \\
&\quad\leq \frac{2}{\mu(0)n}\, \mathbb{Q}\left(F(\widehat{\mathcal{T}}^{(v_m)}, \widehat{U}^{(v_m)})\, \mathbf{1}_{\{|\mathcal{T}|=n+1\}}\right) + e^{-n^{\alpha_0}},
\end{aligned}$$

using Lemma 3.3 and Lemma 4.1 in the last bound. Dividing by $\mathbb{Q}(|\mathcal{T}| = n + 1)$ and using (8), we get

$$\mathbb{Q}^n(F(\mathcal{T}, U)\, \mathbf{1}_{\{\underline{U}>0\}}) \leq \frac{2}{\mu(0)n}\, \mathbb{Q}^n(F(\widehat{\mathcal{T}}^{(v_m)}, \widehat{U}^{(v_m)})) + O(n^{3/2} e^{-n^{\alpha_0}}).$$

By Proposition 4.2, we have $\mathbb{Q}^n(\underline{U} > 0) \geq \widetilde{c}_1/n$. The lemma now follows from the preceding bound, with $\overline{c} = 2/(\mu(0)\widetilde{c}_1)$. □

Proof of Proposition 6.1. *First step.* We first observe that Theorem 2.1 obviously remains valid if $\mathbb{P}_0^n$ is replaced by $\mathbb{Q}^n$. By the Skorokhod representation theorem, we can find, for every integer n sufficiently large, a pair (C_n, V_n) such that the following holds. The processes C_n and V_n are respectively the contour function and the spatial contour function of a spatial tree $(\mathcal{T}_n, U_n)$ with distribution $\mathbb{Q}^n$. Moreover,

$$\left(\left(\frac{\sigma}{2}\ \frac{C_n(2nt)}{n^{1/2}}\right)_{0\le t\le 1}, \left(\frac{1}{\rho}\left(\frac{\sigma}{2}\right)^{1/2} \frac{V_n(2nt)}{n^{1/4}}\right)_{0\le t\le 1}\right) \underset{n\to\infty}{\longrightarrow} (\mathbf{e}, Z^0), \tag{16}$$

uniformly on [0, 1], a.s., and the limiting pair $(\mathbf{e}, Z^0)$ is the Brownian snake with initial point 0, as defined in the introduction above.

In agreement with the previous notation, write v_m^n for the first vertex realizing the minimal spatial position in $\mathcal{T}_n$, and k_n, respectively ℓ_n, for the first, resp. the last, time of visit of v_m^n in the evolution of the contour of $\mathcal{T}_n$.

From Proposition 2.5 in [24], we know that there is a.s. a unique $s_* \in (0, 1)$ such that

$$Z^0(s_*) = \inf_{0\le t\le 1} Z^0(t).$$

The convergence (16) then implies that

$$\lim_{n\to\infty} \frac{k_n}{2n} = \lim_{n\to\infty} \frac{\ell_n}{2n} = s_*, \quad \text{a.s.} \tag{17}$$

Consider then the re-rooted tree $(\widehat{\mathcal{T}}_n^{(v_m^n)}, \widehat{U}_n^{(v_m^n)})$. By construction, its contour function is

$$\widehat{C}_n^{(v_m^n)}(t) = C_n(k_n) + C_n([\![k_n - t]\!]_n) - 2 \inf_{[\![k_n-t]\!]_n \wedge k_n \le r \le [\![k_n-t]\!]_n \vee k_n} C_n(r),$$

for $0 \le t \le 2n - (\ell_n - k_n)$. (Here $[\![k_n - t]\!]_n$ denotes the unique element of $[0, 2n)$ such that $[\![k_n - t]\!]_n - (k_n - t) = 0$ or $2n$.) The corresponding spatial contour function is

$$\widehat{V}_n^{(v_m^n)}(t) = V_n([\![k_n - t]\!]_n) - V_n(k_n).$$

For $2n - (\ell_n - k_n) < t \le 2n$, we also set $\widehat{C}_n^{(v_m^n)}(t) = \widehat{V}_n^{(v_m^n)}(t) = 0$. From (16), (17), and the preceding formulas for $\widehat{C}_n^{(v_m^n)}(t)$ and $\widehat{V}_n^{(v_m^n)}(t)$, we get

$$\left(\left(\frac{\sigma}{2}\ \frac{\widehat{C}_n^{(v_m^n)}(2nt)}{n^{1/2}}\right)_{0\le t\le 1}, \left(\frac{1}{\rho}\left(\frac{\sigma}{2}\right)^{1/2} \frac{\widehat{V}_n^{(v_m^n)}(2nt)}{n^{1/4}}\right)_{0\le t\le 1}\right) \underset{n\to\infty}{\longrightarrow} (\overline{\mathbf{e}}^0, \overline{Z}^0), \tag{18}$$

uniformly on [0,1], a.s., where, as in Section 1,

$$\overline{\mathbf{e}}^0(t) = \mathbf{e}(\{s_* - t\}) + \mathbf{e}(s_*) - 2 \inf_{\{s_*-t\}\wedge s_* \le r \le \{s_*-t\}\vee s_*} \mathbf{e}(r)$$

$$\overline{Z}^0(t) = Z^0(\{s_* - t\}) - Z^0(s_*),$$

where $\{r\}$ denotes the fractional part of r.

Write $\mathbf{P}$ for the probability measure under which the processes (C_n, V_n) and $(\mathbf{e}, Z^0)$ are defined. From Lemma 6.2 applied with a suitable indicator function F, we have for every choice of $\alpha, \delta, \varepsilon > 0$,

$$\begin{aligned}
&\limsup_{n\to\infty} \overline{\mathbb{Q}}^n \left(\left\{ \inf_{t\in[\delta/2,1-\delta/2]} V^{(n)}(t) < 2\alpha \right\} \cup \left\{ \sup_{t\in[0,4\delta]\cup[1-4\delta,1]} (C^{(n)}(t) + V^{(n)}(t)) > \frac{\varepsilon}{2} \right\} \right) \\
&\quad \le \bar{c} \limsup_{n\to\infty} \mathbb{Q}^n \left(\left\{ \inf_{t\in[\delta/2,1-\delta/2]} \widehat{V}^{(v_m),(n)}(t) < 2\alpha \right\} \right.
\end{aligned}$$

$$\cup \left\{ \sup_{t\in[0,4\delta]\cup[1-4\delta,1]} (\widehat{C}^{(v_m),(n)}(t) + \widehat{V}^{(v_m),(n)}(t)) > \frac{\varepsilon}{2} \right\} \Bigg)$$

$$\leq \bar{c}\,\mathbf{P}\left(\left\{ \inf_{t\in[\delta/2,1-\delta/2]} \overline{Z}^0(t) \leq 2\alpha \right\} \cup \left\{ \sup_{t\in[0,4\delta]\cup[1-4\delta,1]} (\overline{\mathbf{e}}^0(t) + \overline{Z}^0(t)) \geq \frac{\varepsilon}{2} \right\} \right), \tag{19}$$

where we used the notation

$$\widehat{C}^{(v_m),(n)}(t) = \frac{\sigma}{2}\, \frac{\widehat{C}^{(v_m)}(2nt)}{n^{1/2}}, \qquad \widehat{C}^{(v_m),(n)}(t) = \frac{1}{\rho}\left(\frac{\sigma}{2}\right)^{1/2} \frac{\widehat{V}^{(v_m)}(2nt)}{n^{1/4}},$$

if $0 \leq t \leq (|\mathcal{T}^{(v_m)}| - 1)/n$, and $\widehat{C}^{(v_m),(n)}(t) = \widehat{V}^{(v_m),(n)}(t) = 0$ if $(|\mathcal{T}^{(v_m)}| - 1)/n < t \leq 1$. In the last inequality above, we used (18) together with the fact that C_n and V_n are respectively the contour function and the spatial contour function of a spatial tree with distribution $\mathbb{Q}^n$.

Recall that $\overline{Z}^0(t) > 0$ for every $t \in (0, 1)$, a.s. Hence, if $b > 0$ and $\varepsilon > 0$ are given, we can first choose $\delta \in (0, \varepsilon)$ so small that

$$\bar{c}\,\mathbf{P}\left(\sup_{t\in[0,4\delta]\cup[1-4\delta,1]} (\overline{\mathbf{e}}^0(t) + \overline{Z}^0(t)) \geq \frac{\varepsilon}{2} \right) < \frac{b}{2}$$

and then find $\alpha \in (0, \varepsilon)$ small enough so that

$$\bar{c}\,\mathbf{P}\left(\inf_{t\in[\delta/2,1-\delta/2]} \overline{Z}^0(t) \leq 2\alpha \right) < \frac{b}{2}.$$

The first part of Proposition 6.1 then follows from (19).

Second step. We now explain how the desired bound under $\overline{\mathbb{P}}_x^n$ can be deduced from the one under $\overline{\mathbb{Q}}$. In a way very similar to the end of the proof of Proposition 4.2, we first choose an integer $\ell \geq 1$ such that $\mu(\ell) > 0$. Then let $y > 0$ be such that $\gamma((y, y+1)) > 0$ and let $p \geq 1$ be the first integer such that $py \geq K$. Also set $m = n + (p-1)\ell + 1$. Under an appropriate conditioning of $\mathbb{Q}$ (requiring in particular that $N_1 = \ell$, $N_{(1,1)} = \ell, \ldots, N_{1^{p-1}} = \ell$), we can embed a tree with distribution $\mathbb{P}_z$, for some random $z \geq K$, into a tree distributed according to $\mathbb{Q}$, and we arrive at the bound

$$\mathbb{Q}\left(\{|\mathcal{T}| = m+1\} \cap \{\underline{U} > 0\} \cap \left\{ \sup_{t\in[0,A+p]\cup[2n-A,2m]} (C(t) + V(t)) \geq \lambda \right\} \right)$$

$$\geq \beta\, \mathbb{P}_x\left(\{|\mathcal{T}| = n+1\} \cap \{\underline{U} > 0\} \cap \left\{ \sup_{t\in[0,A]\cup[2n-A,2n]} (C(t) + V(t)) \geq \lambda \right\} \right), \tag{20}$$

where

$$\beta = \mu(\ell)^{p-1}\, \gamma((y, y+1))^p\, \mu(0)^{(p-1)(\ell-1)}\, \gamma((0,\infty))^{(p-1)(\ell-1)}$$

and $x \in [0, K]$, $A \in (0, 2n)$, $\lambda > 0$ are arbitrary. Similarly, with the same constant β, we have

$$\mathbb{Q}\left(\{|\mathcal{T}| = m+1\} \cap \{\underline{U} > 0\} \cap \left\{ \inf_{t\in[A,2m-A]} V(t) \leq \lambda \right\} \right)$$

$$\geq \beta\, \mathbb{P}_x\left(\{|\mathcal{T}| = n+1\} \cap \{\underline{U} > 0\} \cap \left\{ \inf_{t\in[A,2n-A]} V(t) \leq \lambda - (K+y+1) \right\} \right). \tag{21}$$

Also recall that the quantities $\mathbb{Q}(\{|\mathcal{T}| = m+1\} \cap \{\underline{U} > 0\})$ and $\mathbb{P}_x(\{|\mathcal{T}| = n+1\} \cap \{\underline{U} > 0\})$ are bounded above and below by positive constants times $n^{-5/2}$. Using this last remark, we see that the second part of Proposition 6.1 follows from the first part, (20) and (21). $\square$

Remark. The limiting process $(\overline{\mathbf{e}}^0, \overline{Z}^0)$ in (18) is the same as the one in Theorem 2.2. Therefore, it seems tempting to deduce Theorem 2.2 from (18) and the relations between the conditioned spatial tree and the tree re-rooted at its first minimum (Lemmas 3.3 and 3.4). This approach would indeed be successful, maybe under additional assumptions, in the case when the probability measure γ has no atoms, so that the minimal spatial position is attained at a unique vertex. In our general setting however, we will have to use a different argument which is explained in the next section.

7. Proof of the main result

In this section, we prove Theorem 2.2. We equip $C([0,1],\mathbb{R})^2$ with the norm $\|(f,g)\| = \|f\|_u \vee \|g\|_u$, where $\|f\|_u$ stands for the uniform norm of f. For every $f \in C([0,1],\mathbb{R})$, and every $r > 0$, we also set:

$$\omega_f(r) = \sup_{s,t\in[0,1],|t-s|\le r} |f(t) - f(s)|.$$

We fix $x \geq 0$ and unless otherwise indicated, we argue under $\overline{\mathbb{P}}_x^n$. Let F be a bounded Lipschitz function on $C([0,1],\mathbb{R})^2$. We have to prove that

$$\lim_{n\to\infty} \overline{\mathbb{E}}_x^n[F(C^{(n)}, V^{(n)})] = E[F(\overline{e}^0, \overline{Z}^0)].$$

We may and will assume that $0 \le F \le 1$ and that the Lipschitz constant of F is less than 1. As in Section 1, for every $r > 0$, we denote by $(\mathbf{e}, Z^r)$ a Brownian snake with initial point r and we let $(\overline{\mathbf{e}}^r, \overline{Z}^r)$ be distributed as $(\mathbf{e}, Z^r)$ conditioned on the event

$$\left\{\inf_{0\le t\le 1} Z^r(t) > 0\right\},$$

which has positive probability. We know that

$$(\overline{\mathbf{e}}^r, \overline{Z}^r) \xrightarrow[r\to 0]{\text{(d)}} (\overline{\mathbf{e}}^0, \overline{Z}^0). \tag{22}$$

Lemma 7.1. *Let $0 < c' < c''$. Then,*

$$\lim_{p\to\infty} \sup_{c'p^{1/4}\le y\le c''p^{1/4}} \left| \overline{\mathbb{E}}_y^p \left[F\left(C^{(p)}, V^{(p)}\right)\right] - E\left[F\left(\overline{\mathbf{e}}^{\kappa y/p^{1/4}}, \overline{Z}^{\kappa y/p^{1/4}}\right)\right]\right| = 0$$

where $\kappa = \frac{1}{\rho}(\frac{\sigma}{2})^{1/2}$.

Proof. First note that the law of the infimum of a linear Brownian snake driven by a normalized Brownian excursion $\mathbf{e}$ has no atoms: see Lemma 2.1 in [24] for the case of an unnormalized Brownian excursion $e = (e(t), t \ge 0)$ under the Itô measure, and then use the fact that, for every $\varepsilon > 0$, the law of $(\mathbf{e}(t), 0 \le t \le 1-\varepsilon)$ is absolutely continuous with respect to that of $(e(t), 0 \le t \le 1-\varepsilon)$. It follows that the law of $(\overline{\mathbf{e}}^r, \overline{Z}^r)$ depends continuously on r. It then suffices to show that if (y_p) is a sequence such that $c' p^{1/4} \le y_p \le c'' p^{1/4}$ and $p^{-1/4} y_p \longrightarrow r$, then,

$$\overline{\mathbb{E}}^p_{y_p}[F(C^{(p)}, V^{(p)})] \underset{p\to\infty}{\longrightarrow} E[F(\overline{\mathbf{e}}^{\kappa r}, \overline{Z}^{\kappa r})]. \tag{23}$$

However, Theorem 2.1 implies that

$$\mathbb{E}^p_{y_p}[F(C^{(p)}, V^{(p)})\,\mathbf{1}_{\{\underline{U}>0\}}] \underset{p\to\infty}{\longrightarrow} E[F(\mathbf{e}, Z^{\kappa r})\mathbf{1}_{\{\underline{Z}^{\kappa r}>0\}}]$$

where $\underline{Z}^{\kappa r} = \inf\{Z^{\kappa r}(t),\ 0 \le t \le 1\}$ (we use the fact that $P(\underline{Z}^{\kappa r} = 0) = 0$, as noted above). The desired result (23) readily follows. $\square$

Let $b > 0$. We will prove that for n sufficiently large,

$$|\overline{\mathbb{E}}^n_x[F(C^{(n)}, V^{(n)})] - E[F(\overline{\mathbf{e}}^0, \overline{Z}^0)]| \le 12b,$$

which is enough to get the desired convergence.

By (22), we can choose $\varepsilon \in (0, b \wedge \frac{1}{100})$ small enough so that

$$|E[F(\overline{\mathbf{e}}^r, \overline{Z}^r)] - E[F(\overline{\mathbf{e}}^0, \overline{Z}^0)]| < b \tag{24}$$

for every $0 < r \le 2\varepsilon$. By taking ε smaller if necessary, we can also assume that, for every $r \in (0, 1]$,

$$\begin{aligned}
&E\left[\left(3\varepsilon \sup_{0\le t\le 1} \overline{\mathbf{e}}^r(t)\right) \wedge 1\right] \le b,\\
&E[\omega_{\overline{\mathbf{e}}^r}(6\varepsilon) \wedge 1] \le b,\\
&E\left[\left(3\varepsilon \sup_{0\le t\le 1} \overline{Z}^r(t)\right) \wedge 1\right] \le b,\\
&E\left[\omega_{\overline{Z}^r}(6\varepsilon) \wedge 1\right] \le b.
\end{aligned} \tag{25}$$

For $\delta, \alpha > 0$, denote by $\Gamma_n = \Gamma_n^{\varepsilon,\alpha,\delta}$ the event

$$\Gamma_n = \left\{ \inf_{t\in[\delta,1-\delta]} V^{(n)}(t) \ge \alpha,\ \sup_{t\in[0,3\delta]\cup[1-3\delta,1]} (C^{(n)}(t) + V^{(n)}(t)) \le \varepsilon \right\}.$$

By Proposition 6.1, we can fix $\delta, \alpha \in (0, \varepsilon)$ such that, for every n sufficiently large,

$$\overline{\mathbb{P}}^n_x(\Gamma_n) > 1 - b.$$

On the event Γ_n, we have, for every $t \in [2n\delta, 2n(1-\delta)]$,

$$V(t) \ge \rho \left(\frac{2}{\sigma}\right)^{1/2} \alpha n^{1/4}. \tag{26}$$

Set $\overline{\alpha} = \rho(2/\sigma)^{1/2}\alpha$. The next step of the proof is to apply Corollary 5.2 with $a = \overline{\alpha} n^{1/4}$, assuming that n is large enough so that $a > x$. Let us introduce the relevant notation. We denote by $v_1^n, \ldots, v_{M_n}^n$ the exit vertices from $(-\infty, \overline{\alpha} n^{1/4})$, listed in lexicographical order.

As in Section 5, we can then consider the spatial trees $(\mathcal{T}^{[v_1^n]}, \overline{U}^{[v_1^n]}), \ldots, (\mathcal{T}^{[v_{M_n}^n]}, \overline{U}^{[v_{M_n}^n]})$. The contour functions of these spatial trees may be obtained in the following way. Set

$$k_1^n = \inf\{k \in \mathbb{N} : V(k) \geq \overline{\alpha}\, n^{1/4}\}$$
$$\ell_1^n = \inf\{k \geq k_1^n : C(k+1) < C(k_1^n)\}$$

and, by induction on i,

$$k_{i+1}^n = \inf\{k > \ell_i^n : V(k) \geq \overline{\alpha}\, n^{1/4}\}$$
$$\ell_{i+1}^n = \inf\{k \geq k_{i+1}^n : C(k+1) < C(k_{i+1}^n)\}.$$

Then $k_i^n \leq \ell_i^n < \infty$ iff $i \leq M_n$. The contour function of $\mathcal{T}^{[v_i^n]}$ is

$$(C(k_i^n + t) - C(k_i^n), 0 \leq t \leq \ell_i^n - k_i^n)$$

and the spatial contour function of $(\mathcal{T}^{[v_i^n]}, \overline{U}^{[v_i^n]})$ is

$$(V(k_i^n + t), 0 \leq t \leq \ell_i^n - k_i^n).$$

Note in particular that $U_{v_i^n} = V(k_i^n) = V(\ell_i^n)$, and that $\ell_i^n - k_i^n = 2(|\mathcal{T}^{[v_i^n]}| - 1)$.

By construction, for every integer $k \in [0, 2n] \setminus \cup_{i=1}^{M_n} [k_i^n, \ell_i^n]$, we have $V(k) < \overline{\alpha}\, n^{1/4}$. Also note that $\ell_i^n + 1 < k_{i+1}^n$ for every $i \in \{1, \ldots, M_n - 1\}$.

Using (26), we then see that on the event Γ_n all integer points of $[2n\delta, 2n(1-\delta)]$ must be contained in a single interval $[k_i^n, \ell_i^n]$. Hence, if

$$E_n := \{\exists i \in \{1, \ldots, M_n\} : \ell_i^n - k_i^n > 2(1 - 3\delta)n\}$$

we have $\Gamma_n \subset E_n$ if n is sufficiently large, and so

$$\overline{\mathbb{P}}_x^n(E_n) \geq \overline{\mathbb{P}}_x^n(\Gamma_n) > 1 - b.$$

On the event E_n, we denote by i_n the unique integer $i \in \{1, \ldots, M_n\}$ such that $\ell_i^n - k_i^n > 2(1 - 3\delta)n$. We also set

$$m_n = |\mathcal{T}^{[v_{i_n}^n]}| - 1,$$
$$Y_n = U_{v_{i_n}^n}.$$

Then Corollary 5.2 implies that under the measure $\overline{\mathbb{P}}_x^n(\cdot \mid E_n)$, conditionally on the σ-field

$$\mathcal{G}_n := \sigma\left((\mathcal{T}^{\overline{\alpha} n^{1/4}}, U^{\overline{\alpha} n^{1/4}}), M_n, (|\mathcal{T}^{[v_i^n]}|, 1 \leq i \leq M_n)\right),$$

the spatial tree $(\mathcal{T}^{[v_{i_n}^n]}, \overline{U}^{[v_{i_n}^n]})$ has distribution $\overline{\mathbb{P}}_{Y_n}^{m_n}$. Note that $E_n \in \mathcal{G}_n$ and that Y_n and m_n are $\mathcal{G}_n$-measurable.

Lemma 7.2. *The law of $n^{-1/4} Y_n$ under the measure $\overline{\mathbb{P}}_x^n(\cdot \mid E_n)$ converges as $n \to \infty$ to the Dirac measure at $\overline{\alpha}$.*

Proof. For every $v \in \mathcal{U}^*$, write $\check{v}$ for the father of v. By construction, we have on E_n,

$$Y_n = U_{v_{i_n}^n} \geq \overline{\alpha}\, n^{1/4}$$
$$U_{\check{v}_{i_n}^n} < \overline{\alpha}\, n^{1/4}.$$

To get the statement of the lemma, it thus suffices to verify that, for every $r > 0$,

$$\overline{\mathbb{P}}^n_x\left(\sup_{v\in\mathcal{T}\setminus\{\varnothing\}} \frac{|U_v - U_{\check{v}}|}{n^{1/4}} > r\right) \underset{n\to\infty}{\longrightarrow} 0. \tag{27}$$

If $\overline{\mathbb{P}}^n_x$ is replaced by $\mathbb{P}^n_x$, or by $\mathbb{Q}^n$, (27) becomes a straightforward consequence of (1) (it can also be read from Theorem 2.1). We can then use Lemma 6.2 once again to see that (27) also holds when $\overline{\mathbb{P}}^n_x$ is replaced by $\overline{\mathbb{Q}}^n$. Finally, the same arguments as in the end of the proof of Proposition 4.2 give (27) in the form stated above. □

On the event E_n, we define for $0 \le t \le 1$,

$$\begin{aligned}\widetilde{C}^{(n)}(t) &= \frac{\sigma}{2}\,\frac{C(k^n_{i_n} + 2m_n t) - C(k^n_{i_n})}{m_n^{1/2}}\\ \widetilde{V}^{(n)}(t) &= \frac{1}{\rho}\left(\frac{\sigma}{2}\right)^{1/2} \frac{V(k^n_{i_n} + 2m_n t)}{m_n^{1/4}}.\end{aligned} \tag{28}$$

Note that $\widetilde{C}^{(n)}$ and $\widetilde{V}^{(n)}$ are rescaled versions of the contour function and the spatial contour function of the spatial tree $(\mathcal{T}^{[v^n_{i_n}]}, \overline{U}^{[v^n_{i_n}]})$. On the event E^c_n we take $\widetilde{C}^{(n)}(t) = \widetilde{V}^{(n)}(t) = 0$ for every $0 \le t \le 1$.

By the remarks preceding Lemma 7.2, we have, for any nonnegative measurable function F on $C([0,1],\mathbb{R})^2$,

$$\begin{aligned}\overline{\mathbb{E}}^n_x[1_{E_n} F(\widetilde{C}^{(n)}, \widetilde{V}^{(n)})] &= \overline{\mathbb{E}}^n_x[1_{E_n}\, \overline{\mathbb{E}}^n_x[F(\widetilde{C}^{(n)}, \widetilde{V}^{(n)}) \mid \mathcal{G}_n]]\\ &= \overline{\mathbb{E}}^n_x[1_{E_n}\, \overline{\mathbb{E}}^p_{Y_n}[F(C^{(p)}, V^{(p)})]_{p=m_n}].\end{aligned} \tag{29}$$

We will be able to combine Lemma 7.1 and Lemma 7.2 in order to study the right-hand side of (29). Still we need to explain why $(C^{(n)}, V^{(n)})$ is close to $(\widetilde{C}^{(n)}, \widetilde{V}^{(n)})$ under $\overline{\mathbb{P}}^n_x$, in a suitable sense. Recall the definition of the event $\Gamma_n \subset E_n$, and the fact that $\overline{\mathbb{P}}^n_x(\Gamma^c_n) < b$. Simple estimates show that for all n sufficiently large we have on Γ_n, for every $0 \le t \le 1$,

$$|\widetilde{C}^{(n)}(t) - C^{(n)}(t)| \le \varepsilon + \left(1 - \frac{m_n^{1/2}}{n^{1/2}}\right) \sup_{0\le s\le 1} \widetilde{C}^{(n)}(s) + \omega_{\widetilde{C}^{(n)}}(6\delta).$$

In the previous inequality, we used the bounds

$$n \ge m_n \ge (1-3\delta)n, \qquad k^n_{i_n} < 3\delta n$$

that hold on Γ_n for n large. Since

$$1 - \frac{m_n^{1/2}}{n^{1/2}} \le 1 - \frac{m_n}{n} \le 3\delta,$$

we finally get on Γ_n

$$\sup_{0\le t\le 1} |\widetilde{C}^{(n)}(t) - C^{(n)}(t)| \le \varepsilon + 3\delta \sup_{0\le t\le 1} \widetilde{C}^{(n)}(t) + \omega_{\widetilde{C}^{(n)}}(6\delta). \tag{30}$$

Similarly, we have on Γ_n

$$\sup_{0\le t\le 1} |\widetilde{V}^{(n)}(t) - V^{(n)}(t)| \le \varepsilon + 3\delta \sup_{0\le t\le 1} \widetilde{V}^{(n)}(t) + \omega_{\widetilde{V}^{(n)}}(6\delta). \tag{31}$$

Let us now complete the proof. By Lemma 7.1, if p is sufficiently large, we have

$$\sup_{\frac{\overline{\alpha}}{2}p^{1/4}\leq y\leq 2\overline{\alpha}p^{1/4}}\left|\overline{\mathbb{E}}^p_y\left[F\left(C^{(p)},V^{(p)}\right)\right]-E\left[F\left(\overline{\mathbf{e}}^{\kappa y/p^{1/4}},\overline{Z}^{\kappa y/p^{1/4}}\right)\right]\right|<b.$$

Since $2\kappa\overline{\alpha}=2\alpha<2\varepsilon$, we can combine this with (24) to get

$$\sup_{\frac{\overline{\alpha}}{2}p^{1/4}\leq y\leq 2\overline{\alpha}p^{1/4}}|\overline{\mathbb{E}}^p_y[F(C^{(p)},V^{(p)})]-E[F(\overline{\mathbf{e}}^0,\overline{Z}^0)]|<2b.$$

Now recall (29), the fact that $m_n\geq(1-3\delta)n$ on E_n and Lemma 7.2. It follows that for n sufficiently large,

$$|\overline{\mathbb{E}}^n_x[1_{E_n}F(\widetilde{C}^{(n)},\widetilde{V}^{(n)})]-\overline{\mathbb{P}}^n_x(E_n)\,E[F(\overline{\mathbf{e}}^0,\overline{Z}^0)]|<3b.$$

Since $\overline{\mathbb{P}}^n_x(E_n)>1-b$, this implies

$$|\overline{\mathbb{E}}^n_x[F(\widetilde{C}^{(n)},\widetilde{V}^{(n)})]-E[F(\overline{\mathbf{e}}^0,\overline{Z}^0)]|<4b. \tag{32}$$

Furthermore, from the bounds (30) and (31), we have

$$\begin{aligned}&\overline{\mathbb{E}}^n_x[1_{\Gamma_n}|F(\widetilde{C}^{(n)},\widetilde{V}^{(n)})-F(C^{(n)},V^{(n)})|]\\&\quad\leq 2\varepsilon+\overline{\mathbb{E}}^n_x\left[\left(3\delta\sup_{0\leq t\leq 1}\widetilde{C}^{(n)}(t)\right)\wedge 1+\omega_{\widetilde{C}^{(n)}}(6\delta)\wedge 1\right]\\&\qquad+\overline{\mathbb{E}}^n_x\left[\left(3\delta\sup_{0\leq t\leq 1}\widetilde{V}^{(n)}(t)\right)\wedge 1+\omega_{\widetilde{V}^{(n)}}(6\delta)\wedge 1\right].\end{aligned}$$

At this point, we can again use (29), Lemmas 7.1 and 7.2 to see that the right-hand side is bounded above for n sufficiently large by

$$\begin{aligned}&b+2\varepsilon+\sup_{0<r\leq 1}E\left[\left(3\varepsilon\sup_{0\leq t\leq 1}\overline{\mathbf{e}}^r(t)\right)\wedge 1+\omega_{\overline{\mathbf{e}}^r}(6\varepsilon)\wedge 1\right]\\&\quad+\sup_{0<r\leq 1}E\left[\left(3\varepsilon\sup_{0\leq t\leq 1}\overline{Z}^r(t)\right)\wedge 1+\omega_{\overline{Z}^r}(6\varepsilon)\wedge 1\right].\end{aligned}$$

From (25), the latter quantity is bounded above by $7b$. Since $\overline{\mathbb{P}}^n_x(\Gamma_n)>1-b$, this gives the bound

$$\overline{\mathbb{E}}^n_x[|F(\widetilde{C}^{(n)},\widetilde{V}^{(n)})-F(C^{(n)},V^{(n)})|]\leq 8b.$$

Combining this bound with (32) leads to

$$|\overline{\mathbb{E}}^n_x[F(C^{(n)},V^{(n)})]-E[F(\overline{\mathbf{e}}^0,\overline{Z}^0)]|\leq 12b,$$

which completes the proof of Theorem 2.2. □

8. An application to random quadrangulations

In this section, we apply Theorem 2.2 to give a short derivation of some asymptotics for random quadrangulations which were obtained in [8]. Let us briefly recall the main definitions, following Section 2 of [8]. A *planar map* is a proper embedding, without edge crossings, of a connected graph in the plane. Loops and multiple edges are allowed. A planar map is *rooted* if there is a distinguished edge on the border of the infinite face, which is called the root edge. By convention, the root edge is oriented counterclockwise, and its origin is called the root vertex. The set of vertices will always be equipped with the graph distance: if a and a' are two vertices, $d(a, a')$ is the minimal number of edges on a path from a to a'. Two rooted planar maps are said to be equivalent if there exists a homeomorphism of the plane that sends one map onto the other one and preserves the root edges.

A planar map is a *quadrangulation* if all faces have degree 4. A quadrangulation contains no loop but may contain multiple edges. For every integer $n \geq 2$, we denote by $\mathcal{Q}_n$ the set of all (equivalent classes of) quadrangulations with n faces. Then $\mathcal{Q}_n$ is a finite set, whose cardinality was computed by Tutte [31]:

$$|\mathcal{Q}_n| = \frac{2}{n+2}\,\frac{3^n}{n+1}\binom{2n}{n}.$$

The relations between planar maps and the present work come from a basic result [10,30] connecting quadrangulations with the so-called well-labelled trees. Let us call a labelled tree any spatial tree $(\mathcal{T}, U)$ such that $U_\varnothing = 1$, $U_v \in \mathbb{Z}$ for every $v \in \mathcal{T}$ and $|U_v - U_{\check{v}}| \leq 1$ for every $v \in \mathcal{T} \setminus \{\varnothing\}$ (recall that $\check{v}$ is the father of v). The tree is said to be well-labelled if in addition $U_v \geq 1$ for every $v \in \mathcal{T}$. We denote by $\mathbb{T}_n$ the collection of all labelled trees with $n+1$ vertices, and by $\mathbb{T}_n^0$ the collection of all well-labelled trees with $n+1$ vertices.

Theorem 8.1. *There exists a bijection Φ_n from $\mathcal{Q}_n$ onto $\mathbb{T}_n^0$, which enjoys the following additional property. Let $q \in \mathcal{Q}_n$ and $(\mathcal{T}, U) = \Phi_n(q)$. Then, if $\mathcal{V}_q$ denotes the set of vertices of q, and $a_0 \in \mathcal{V}_q$ is the root vertex of q, we have for every integer $k \geq 1$:*

$$|\{a \in \mathcal{V}_q : d(a_0, a) = k\}| = |\{v \in \mathcal{T} : U_v = k\}|.$$

See Section 3 of [8] for a detailed proof. Assume that $q \in \mathcal{Q}_n$ and $(\mathcal{T}, U) \in \mathbb{T}_n^0$ are such that $(\mathcal{T}, U) = \Phi_n(q)$. Then one can construct a one-to-one correspondence between vertices a of q other than the root vertex, and vertices v of the tree $\mathcal{T}$, in such a way that the distance $d(a_0, a)$ from the root coincides with the spatial position U_v (see [8] for details). This explains the final formula of the theorem.

Before stating the main asymptotic result, let us introduce the relevant notation. If q is a rooted quadrangulation, the radius $r(q)$ is the maximal distance between the root vertex a_0 and another vertex a. The profile λ_q is the integer-valued measure on $\mathbb{N}$ defined by

$$\lambda_q(k) = |\{a \in \mathcal{V}_q : d(a_0, a) = k\}|.$$

Note that $r(q)$ is just the supremum of the support of λ_q. It is also convenient to introduce the rescaled profile. If $q \in \mathcal{Q}_n$, this is the probability measure on $\mathbb{R}_+$ defined by

$$\lambda_q^{(n)}(A) = \frac{1}{n+1}\,\lambda_q(n^{1/4}A)$$

for any Borel subset A of $\mathbb{R}_+$.

Theorem 8.2. (i) *The law of $n^{-1/4} r(q)$ under the uniform probability measure on $\mathcal{Q}_n$ converges as $n \to \infty$ to the law of the variable*

$$\left(\frac{8}{9}\right)^{1/4}\left(\sup_{0\leq s\leq 1} Z^0(s) - \inf_{0\leq s\leq 1} Z^0(s)\right).$$

(ii) *The law of the random measure $\lambda_q^{(n)}$ under the uniform probability measure on $\mathcal{Q}_n$ converges as $n \to \infty$ to the law of the random probability measure $\mathcal{I}$ defined by*

$$\langle \mathcal{I}, g \rangle = \int_0^1 \mathrm{d}r \; g\left(\left(\frac{8}{9}\right)^{1/4} (Z^0(r) - \inf_{0 \le s \le 1} Z^0(s)) \right).$$

(iii) *The law of the rescaled distance $n^{-1/4} d(a_0, a)$ from a vertex a chosen uniformly at random among all vertices of q to the root vertex a_0, under the uniform probability measure on $\mathcal{Q}_n$, converges as $n \to \infty$ to the law of the random variable*

$$\left(\frac{8}{9}\right)^{1/4} \left(\sup_{0 \le s \le 1} Z^0(s) \right).$$

Remarks. (a) Part (i) of the theorem is in Corollary 3 of [8] (which also gives the convergence of moments). Part (ii) is Corollary 4 of [8]. Part (iii) is not stated in [8], but as we will see it is a straightforward consequence of (ii).

(b) We could also have given the various limits in Theorem 8.2 in terms of the random measure known as (one-dimensional) ISE. Up to the trivial multiplicative constant $(8/9)^{1/4}$, the limit in (i) is the length of the support of ISE, the limit in (iii) is the supremum of this support, and the random measure $\mathcal{I}$ appearing in (ii) is ISE itself shifted by the minimum of its support. As is justified in [24], this shifting is equivalent to conditioning ISE to be supported on the positive half-line.

(c) Detailed information about the limiting laws in (i) and (iii) can be found in [11] and in the recent preprint [2].

Proof. We apply the results of the preceding sections taking $\mu(k) = 2^{-k-1}$ for $k \in \mathbb{Z}_+$ and letting γ be the uniform probability measure on $\{-1, 0, 1\}$: $\gamma(-1) = \gamma(0) = \gamma(1) = \frac{1}{3}$. Note that we have then $\sigma^2 = 2$, $\rho^2 = 2/3$ and thus $\kappa = \frac{1}{\rho} (\frac{\sigma}{2})^{1/2} = (\frac{9}{8})^{1/4}$.

With the preceding choice of μ and γ, one immediately verifies that $\mathbb{P}_1^n$ is the uniform probability measure on $\mathbb{T}_n$, and $\overline{\mathbb{P}}_1^n$ is the uniform probability measure on $\mathbb{T}_n^0$. The various assertions of Theorem 8.2 can then be obtained by combining Theorem 8.1 with Theorem 2.2.

To begin with, Theorem 8.1 entails that the law of $r(q)$ under the uniform probability measure on $\mathcal{Q}_n$ coincides with the law of $\sup\{U_v : v \in \mathcal{T}\}$ under $\overline{\mathbb{P}}_1^n$. Since by construction, if $(\mathcal{T}, U) \in \mathbb{T}_n$,

$$\sup\{U_v : v \in \mathcal{T}\} = \sup\{V(t) : t \in [0, 2n]\}$$

Theorem 2.2 readily implies that the law of $n^{-1/4} \sup\{U_v : v \in \mathcal{T}\}$ under $\mathcal{Q}_n$ converges to the law of

$$\left(\frac{8}{9}\right)^{1/4} \left(\sup_{0 \le s \le 1} \overline{Z}^0(s) \right).$$

From the "Vervaat transformation" connecting the conditioned Brownian snake and the unconditioned one (cf. Section 1), this is the same as the limit in (i).

Let us turn to (ii). By Theorem 8.1, the law of $\lambda_q^{(n)}$ under the uniform probability measure on $\mathcal{Q}_n$ coincides with the law under $\overline{\mathbb{P}}_1^n$ of the random measure $\mathcal{I}_n$ defined by

$$\langle \mathcal{I}_n, g \rangle = \frac{1}{n+1} \sum_{v \in \mathcal{T}} g(n^{-1/4} U_v).$$

In view of our asymptotics, we may replace $\mathcal{I}_n$ by $\mathcal{I}'_n$ defined by

$$\langle \mathcal{I}'_n, g\rangle = \frac{1}{n} \sum_{v\in\mathcal{T}\setminus\{\varnothing\}} g(n^{-1/4}U_v).$$

Now, from the definition of the contour function C and of the spatial contour function V, it is elementary to verify that we have also

$$\langle \mathcal{I}'_n, g\rangle = \frac{1}{2n} \int_0^{2n} \mathrm{d}t\, g\left(\frac{V([t]_C)}{n^{1/4}}\right)$$

where if $t \in [k, k+1)$ we set $[t]_C = k$ if $C(k) \geq C(t)$ and $[t]_C = k+1$ otherwise. In this form, and using the fact that $|[t]_C - t| \leq 1$, we deduce from Theorem 2.2 that the law of $\mathcal{I}'_n$ under $\overline{\mathbb{P}}^n_1$ converges to the law of the random measure $\mathcal{I}'$ defined by

$$\langle \mathcal{I}', g\rangle = \int_0^1 \mathrm{d}r\, g\left(\left(\frac{8}{9}\right)^{1/4} \overline{Z}^0(r)\right).$$

Again the Vervaat transformation shows that this is the same as the limit in (ii).

Finally, let X_n be distributed as $n^{-1/4}d(a_0, a)$ when the quadrangulation q is uniform over $\mathcal{Q}_n$ and a is uniform over the set of vertices of q other than the root vertex a_0, and let g be bounded and continuous on $\mathbb{R}_+$. Then,

$$E[g(X_n)] = \frac{1}{|\mathcal{Q}_n|} \sum_{q\in\mathcal{Q}_n} \int \lambda_q^{(n)}(\mathrm{d}x)\, g(x).$$

From (ii), this converges towards

$$E\left(\int_0^1 \mathrm{d}r\, g\left(\left(\frac{8}{9}\right)^{1/4} (Z^0(r) - \inf_{0\leq s\leq 1} Z^0(s))\right)\right).$$

Now, by the invariance property of the Brownian snake under uniform re-rooting (see e.g. Theorem 2.3 in [24]), the latter quantity is equal to

$$E\left(g\left(-\left(\frac{8}{9}\right)^{1/4} \inf_{0\leq s\leq 1} Z^0(s)\right)\right) = E\left(g\left(\left(\frac{8}{9}\right)^{1/4} \sup_{0\leq s\leq 1} Z^0(s)\right)\right),$$

by symmetry. This completes the proof. $\square$

Let us conclude with some remarks. The cardinality of $\mathbb{T}_n$ is 3^n times the cardinality of the set of rooted ordered trees with $n+1$ vertices, which is the Catalan number of order n:

$$|\mathbb{T}_n| = \frac{3^n}{n+1}\binom{2n}{n}.$$

Comparing with the formula for $|\mathcal{Q}_n| = |\mathbb{T}_n^0|$, we see that

$$\mathbb{P}_1^n(\underline{U} > 0) = \frac{|\mathbb{T}_n^0|}{|\mathbb{T}_n|} = \frac{2}{n+2}$$

(cf. Theorem 2 in [8] for a combinatorial explanation). This is of course consistent with the estimates of Proposition 4.2.

The proofs in [8] are based on a form of Theorem 2.1 (which allows one to deal with labelled trees) and some delicate combinatorial arguments that are needed to relate well-labelled trees with labelled trees (the latter are called embedded trees in [8]). The originality of our approach is thus to apply asymptotics for well-labelled trees, viewed here as conditioned trees, rather than to use a combinatorial method to get rid of the conditioning. We expect that this method will have applications to other types of planar maps, which are also known to be in one-to-one correspondence with various classes of discrete trees (see [6] and [25]).

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